MEASURING PERSPECTIVE OF FRACTION KNOWLEDGE: INTEGRATING HISTORICAL AND NEUROCOGNITIVE FINDINGS[*]

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Abstract

In this theoretical investigation, we describe the origins and cognitive difficulties involved in the common conception of fractional numbers, where a fraction corresponds to some parts of an equally partitioned whole. As an alternative, we present a new notion of fraction knowledge, called the perspective of measure-proportionality. It is informed both by the historical-cultural analysis of the emergence of fractions in social practice and by neuroscientific evidence of the propensity of human beings to perceive from childhood nonsymbolic proportionality between pairs of quantities. We suggest that this natural neurocognitive propensity of individuals may be an instructional link to develop students' robust knowledge about fractional numbers.

Keywords: Fraction knowledge, Measuring perspective, Nonsymbolic fraction ideas

Resumo

Nesta investigação teórica, descrevemos as origens e dificuldades cognitivas implicadas na concepção comum de números fracionários, em que uma fração corresponde a umas partes de um todo dividido em partes iguais. Como uma alternativa, apresentamos uma nova noção de conhecimento de fração, chamada a perspectiva de medindo-proporcionalidade. Ela é informada tanto pela análise histórico-cultural do surgimento de frações na prática social quanto pela evidência neurocientífica da propensão dos seres humanos, desde a infância, a perceber a proporcionalidade não-simbólica entre pares de quantidades. Sugerimos que essa propensão neurocognitiva natural dos indivíduos possa ser um elo instrucional para desenvolver o conhecimento robusto dos estudantes sobre números fracionários.

Palavras-chave: Conhecimento de fração, Perspectiva de medindo, Ideias de frações não-simbólicas

INTRODUCTION

Algebra is the gateway to higher mathematics; however, the gate's key is fraction[1] knowledge. Policy makers, psychologists, mathematicians, and mathematics education

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[1] In this article, we focus specifically on positive fractions; that is, positive rational numbers represented in the form $\frac{p}{q}$, where p and q are positive natural numbers, $\{1, 2, 3, \dots\}$.

researchers alike recognize both anecdotally and empirically that conceptual knowledge of fractions and, more generally, rational numbers constitute a necessary condition for successful participation and high performance in advanced mathematics including algebra, probability, statistics, and calculus (BAILEY et al., 2012; BOOTH; NEWTON, 2012; LAMON, 2007; 2012; MAHER; YANKELEWITZ, 2017; MATTHEWS; ZIOLS, 2019; NATIONAL MATHEMATICS ADVISORY PANEL, 2008; SIEGLER et al., 2012; TORBEYNS et al., 2015; WU, 2001; 2009). Nevertheless, students worldwide have difficulties comprehending fractions and operating with them (OECD, 2014). Furthermore, competence with arithmetic, which includes fractions, contributes in adulthood to employment, wage, and salary opportunities (RITCHIE; BATES, 2013). Nevertheless, in the United States, high achievement in mathematics lags among both students and teachers, owing mainly to their lack of conceptual knowledge of fractions and operations on them (LAMON, 2007; LIN et al., 2013). Learners' conceptual difficulties cause them to order fractions and operate on them incorrectly as well as not to conceive of fractions as quantities representing magnitudes and as a dense subset of the real numbers (BEHR et al., 1984; NI; ZHOU, 2005; SIEGLER, 2016). Another source for the challenge to understand fractions is the whole-number or natural-number bias (BEHR et al., 1983; GÓMEZ et al., 2015; NI; ZHOU, 2005; VAMVAKOUSSI; VAN DOOREN; VERSCHAFFEL, 2012). That is, the tendency to apply inappropriately properties of natural numbers to fraction tasks. For example, students may judge 4/7 to be bigger than 2/3 since as whole numbers $4 > 2$ and $7 > 3$ (GÓMEZ et al., 2015). Finally, researchers note that students have difficulty with the concept of unit, a fundamental element for the cognitive construction of fractions (CAMPOS; RODRIGUES, 2007). These documented problematic understandings are configured by current dominant approaches to fraction instruction.

In the United States, fraction instruction and consequent student understanding center on a specific view of the nature of fractions and its associated definitional interpretation—the partitioning perspective [2] (see, for example, BEHR et al., 1992;

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[2] Recently, Nilce Fatima Scheffer, Universidade Federal da Fronteira Sul (UFFS), and I verified that this perspective also pervades how fractions are introduced in the 14 approved textbooks of the 2019 Brazilian Programa Nacional do Livro Didático (PNLD). See SCHEFFER e POWELL (in press). The part/whole interpretation is also prevalent in Spain (ESCOLANO VIZCARRA; GAIRÍN SALLÁN, 2005). It is also important to note that the majority of fraction problems in textbooks require procedural rather than conceptual knowledge (SON; SENK, 2010).

KERSLAKE, 1986; TZUR, 1999). This definitional interpretation, called the "part/whole" conception, defines fractions as discrete, countable parts of an equipartitioned whole (e.g., slices of a pizza) and, as LAMON (2001) emphasizes, is "mathematically and psychologically … not sufficient as a foundation for the system of rational numbers" (p. 150). Though researchers and educators alike know that the part/whole and, more generally, the partitioning perspective is epistemologically deficient, what is little known is how other perspectives on the nature of fractions and definitional views may facilitate learners to develop a robust conceptual understanding of fractions. Absence a re-examination and re-formulation of the nature of fractions knowledge, ineffective instructional practices about fractions will continue to limit learners' participation in advance mathematics and related disciplines, especially of individuals from racial, ethnic, gender, and economic groups already underrepresented generally in scientific fields.

An alternative perspective of fraction knowledge is what we call a measuring interpretation. It is informed by both a historical-cultural analysis of the origins of fractions (ALEKSANDROV, 1963; COURANT; ROBBINS, 1941/1996; GILLINGS, 1972/1982; ROQUE, 2012) as well as cognitive and neuroscientific findings about ratios as precepts (LEWIS; MATTHEWS; HUBBARD, 2015; MATTHEWS; ELLIS, 2018; SIEGLER et al., 2013; SIEGLER; LORTIE-FORGUES, 2014). Based on our measuring interpretation, we propose an alternative theoretical view and suggest a new epistemological pathway for fraction knowledge. The significance of this pathway is twofold. First, it corresponds to human's cognitive propensity from infancy to discern nonsymbolic ratios between pairs of quantities. Second, by basing the pathway on the historical source of fractions, it addresses documented problematic understandings fractions and promises to enable learners to construct a robust understanding of fraction magnitude. Understanding magnitude links numerical development from whole numbers to rational (and irrational) numbers. Concerning this development, SIEGLER e LORTIE-FORGUES (2014) suggest that "[t]he developmental process includes at least four trends: representing nonsymbolic numerical magnitudes increasingly precisely, linking nonsymbolic and symbolic representations of small whole numbers, extending the range of numbers whose magnitudes are accurately represented to larger whole numbers, *and* representing accurately the magnitudes of rational numbers, including fractions, decimals, percentages, and negatives" (p. 148, emphasis added). Moreover, the

magnitude of fractional numbers is at the core of our theorization of fraction sense (POWELL; ALI, 2018).

In this theoretical investigation, we sketch the historical origins of fractions and the emergence of the partitioning interpretation as well as cognitive difficulties that this interpretation entails. Afterward, we propose an alternative view—the measuring perspective—and discuss how learners can construct first nonsymbolically and later symbolically ideas about fractions and their magnitudes.

HISTORICAL-CULTURAL PERSPECTIVE OF THE SOCIAL GENESIS OF FRACTIONS

Conceptual views of mathematical objects have historical sources that in turn shape those objects' ontological and epistemological perspectives. Understanding the nature or ontology of a mathematical object commences with its historical origin. Awareness of the object's genesis influences perspectives on how one acquires knowledge of it or its epistemology. Fraction knowledge is a case in point.

More than four millennia ago, in Mesopotamian and Egyptian cultures, along the Tigris, Euphrates and Nile rivers, with the birth of agriculture, material conditions introduced the need to invent cognitive ways to measure quantities of land, crops, seeds, and so forth and to record the measures (CLAWSON, 1994/2003; STRUIK, 1948/1967). For example, to measure continuous quantities such as the distances of land, ancient surveyors stretched ropes, in which the length between two nodes represented a unit of measure. In this social practice of measuring lengths as well as areas and volumes arose simultaneously geometry and fractional numbers (ALEKSANDROV, 1963; CARAÇA, 1951; ROQUE, 2012). These measuring practices were part of the social life of ancient Egypt. Egyptians employed these practices to construct pyramids more than 1000 years before (or 5000 years ago) the famous Ahmes and Moscow papyri were scribed (RESNIKOFF; WELLS JR., 1973/1984; STRUIK, 1948/1967).

Focusing on fractions, ontologically, they emerged to know, for instance, the extent of a distance or length, d, in comparison to a unit of measure, u. The length equals an integral multiple, a, of u and possibly an additional amount, r, that is less than u, $d = au + r$. The additional amount come to be represented as a ratio of the remaining amount to the unit of measure, r/u. Later, the ancient Greeks discovered that such ratios

were not always commensurable (STRUIK, 1948/1967). As an ontological consequence, a fraction can be defined as *a multiplicative comparison between two commensurable quantities*.

For more than three or four millennia, the use and notation of fractions evolved. Only in the 17th century did mathematicians accept fractions as objects on a par with natural numbers. The acceptance of fractions permitted equations of the form $ax = b$ to have solutions, $x = b/a$, without restriction, provided that $a \neq 0$. It also allowed for the generalization of numbers with the four operations—addition, subtraction, multiplication, and division—to be a closed domain, an algebraic field (COURANT; ROBBINS, 1941/1996). The acceptance of fractions to allow for the division of two natural numbers, where the divisor is nonzero, leads to equating a fraction to equipartitioning an object (DAVYDOV; TSVETKOVICH, 1991).

EMERGENCE OF THE PARTITIONING INTERPRETATION

The theoretical expansion of the domain of numbers to include fractions bestowed meaningfulness upon the result of the division of natural numbers when the divisor is not a factor of the dividend. Besides, by the late 16th century CE, Simon Stevin of Bruges had already written a systematic treatment of both common fractions and decimal fractions in his book, *De Thiende* (*The Tenth*) (FLAGG, 1983). Therefore, in the 17th century, when fractions finally were accepted, rational numbers already had two symbolic representations.

As for fractions, neither their symbolic representation nor their theoretical justification as number proved sufficient or even epistemologically desirable to support learners' psychological acceptance and understanding. As such, learners' mental representations of fraction needed support. On this point, DAVYDOV e TSVETKOVICH (1991) note the following:

> Fifth graders, and younger school children even more so, cannot be given the principle of that division which leads to fractions in a pure symbolic form. Its visual correlate had to be found. It is in this role that the so-called division of things themselves appeared, their subdivision into parts which in the course of teaching can be relatively easily tied to terms characteristic for defining ordinary fractions. (p. 24)

They argue that the ontological stance that fractions emerge from the division of natural numbers becomes associated epistemologically with the physical and visual division of objects. These two correlates connect symbolic representations of fractions with images such as the physical division of the areas of circles or rectangles into equal regions. The fraction a/b is then defined visually as a equal regions of the area of a circle or rectangle divided into b of those regions.

The need for physical and visual correlates of fractions is the origin of the partitioned or part/whole interpretation of fractions (DAVYDOV; TSVETKOVICH, 1991). It has become instructionally privileged from among KIEREN'S (1976; 1988) various interpretations of a fraction. Nevertheless, for students, this interpretation can be epistemologically problematic. Consider what the shaded regions represent in the two circles in Figure 1.
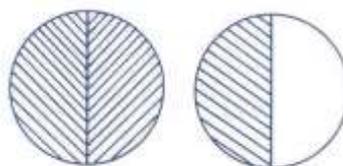

Figure 1 – What fraction do the shaded portions represent?
Source: GATTEGNO; HOFFMAN, 1976, p. IA5

By presenting this standard illustration for $\frac{3}{2}$ in Figure 1, GATTEGNO e HOFFMAN (1976) question whether students can be faulted for concluding that the shaded regions represent $\frac{3}{4}$ of 1 or even $\frac{3}{2}$ of 2 without knowing what is considered to be the whole or the unit? Students who only work with visual models of things partitioned, may develop limited strategies such as counting the number of pieces rather than assessing a multiplicative relationship between two quantities. Moreover, conceiving of fractions as "parts of a whole," students have difficulty making sense of fractions whose numerator is larger than the denominator such as $\frac{7}{4}$ and conceding that a fraction is a number, not just parts of something (TUCKER, 2008).

NEUROCOGNITIVE RESULTS

Distinct from the ontological perspective of fraction arising from the equal division or partitioning of things, our fraction perspective has two primary sources. We

have already presented the emergence of fractions with ontological roots in the historical, social practice of measuring. Our measuring perspective is also founded on recent findings in cognitive science and neuroscience that view ratios as precepts (LEWIS; MATTHEWS; HUBBARD, 2015; MATTHEWS; ELLIS, 2018). Infants as young as 6-months old are capable of discerning two nonsymbolic ratios whose values are sufficiently far apart, years before they learn formally about proportionality in school (MCCRINK; WYNN, 2007). Based on neuroscientific evidence that a population of neurons encode fraction numerals (e.g., 3/6) or words (e.g., one-half) by their numerical magnitude and not necessarily separately by numerator and denominator (see, for example, ISCHEBECK; SCHOCKE; DELAZER, 2009; JACOB; NIEDER, 2009a; b), MATTHEWS e ELLIS (2018) posit that "human beings have intuitive, perceptually-based access to primitive ratio concepts when they are instantiated using nonsymbolic graphical representations" (p. 23). Infants as young as six-months old, pre-school children, as well as young adults are able among visual representations to recognize and compare accurately ratios of nonsymbolic objects (DUFFY; HUTTENLOCHER; LEVINE, 2005; LEWIS; MATTHEWS; HUBBARD, 2015; MCCRINK; WYNN, 2007; SOPHIAN, 2000). This perceptually-based neurocognitive ability to discriminate nonsymbolic ratios has been term by LEWIS; MATTHEWS e HUBBARD (2015) as the ratio processing system (RPS). MATTHEWS e CHESNEY (2015) question how the RPS can be leveraged for fraction instruction. Furthermore, MATTHEWS e ELLIS (2018) argue that KIEREN'S (1976; 1988) list of fraction interpretations should be augmented to include the interpretation of "rational numbers as ratios of nonsymbolic quantities" (p. 24).

Humans not only innately perceive ratios of nonsymbolic quantities but also process their magnitudes in the same neural region where they represent magnitudes of symbolic proportions (OBERSTEINER et al., 2019). Employing functional magnetic resonance imaging (fMRI) to measure regional brain activity, MOCK et al. (2018) have found a shared neural substrate that processes relative magnitudes of both nonsymbolic and symbolic ratios. Specifically, they observed that specific occipito-parietal areas including right intraparietal sulcus (IPS) are engaged during proportion magnitude processing (see Figure 2). As such, their finding suggests that pedagogic practices that present the two types of ratios relationally may be efficacious.

Figure 2. Areas of joint neural activation across conditions involving symbolic and nonsymbolic fractions. These areas are contained within the intraparietal sulcus (IPS) brain region.
Source: MOCK et al. (2018), page 13.

Aside from locating regions of the human brain involved in discerning nonsymbolic and symbolic ratios, neuroscientific investigations conclude that inhibitory processes play a significant role in fraction comparison. Executive functions are employed in complex multistep processes and goal-directed problem solving and, therefore, common in doing mathematics. Among these mental functions, crucial for mathematical competence is the ability to inhibit, stop, or override prepotent or automatized mental responses (GÓMEZ et al., 2015). Some researchers operationalize and measure inhibition reaction time and accuracy by using a Stroop task, where two sources of unrelated information compete for a subject's attention. GÓMEZ et al. (2015) used a numerical Stroop task, where research participants choose one of two single-digit numbers presented on a computer screen that has the greater numerical magnitude. The competing information was the magnitude of the single-digit number's font. Their Stroop task contains three conditions: (1) *congruent* items are those in which numerical magnitude and physical size are both maximized by the same digit (e.g., 3 vs. 7); (2) *incongruent* items are those in which one digit is numerically greater, but the other digit is physically larger (e.g., 3 vs. 7); and (3) *neutral* items are those in which both digits have the same physical size and hence the comparison between them has no competing or distracting information other than numerical magnitude (e.g., 3 vs. 7) (GÓMEZ et al., 2015, p. 802). GÓMEZ et al. (2015) found that middle school students with stronger inhibitory control are more likely to be proficient in fraction comparison. They are more likely to reason beyond natural number bias when comparing fractions. This result suggests that instructional practices that rather than teach learners about fractions as compositional entities of two natural numbers prime learners to attend to fractions as a unitary entity with magnitude.

MEASURING PERSPECTIVE

Employing this pedagogical insight, recognizing the common neural correlates of symbolic and nonsymbolic fractions, and incorporating how MATTHEWS e ELLIS (2018) interpret a fraction as also being a nonsymbolic quantity, our measuring perspective of fraction knowledge consists of two components. It begins with nonsymbolic fractions and progresses to symbolic fractions (POWELL, 2018a; b). We provide opportunities for students to interact with visible and tangible, commensurable and continuous objects or quantities, first to develop a language to describe the multiplicative comparative relation among pairs of quantities that they discern, and second to practice articulating that language so that they are comfortable verbalizing the multiplicative comparisons among pairs of the objects. Afterward, without the aid of the physical objects, students imagine them and talk about the multiplicative relations among pairs of the objects. Later, once they have the facility with stating relations among imagined pairs of quantities, to record their statements, the students learn to use mathematical, symbolic notations. These instructional phases are three of four phase of the 4A-Instructional Model as described in POWELL (2018b).

We now illustrate our measuring perspective. We use Cuisenaire rods, a simple but inventive collection of physical materials (wooden parallelepipeds) or manipulatives with which learners can quickly become familiar (see Figure 3). To become familiar with the rods and relations among them, learners need to engage in both free play and structured tasks in which they attend to the tangible (length) and visible (color) characteristics of the rods. We have learners focus their attention on the length of rods as the measurable attribute about which to construct multiplicative comparisons between quantities. Cuisenaire rods consist of ten different sizes and colors (see Figure 3). Rods of the same color have the same length and *vice versa* and the length of each color rod in sequence—white, red, green, purple, yellow, dark-green, ebony, tan, blue, and orange—increases by one centimeter, from 1 to 10 centimeters long. Because of their simplicity, while students work on mathematics tasks, Cuisenaire rods do not generate high extraneous cognitive load (SWELLER, 1994; SWELLER; VAN MERRIENBOER; PAAS, 1998). As such, they allow learners to focus mainly on becoming aware of relations among the rods, which yield ideas about whole numbers, fractions, and operations on them.

Figure 3 – Cuisenaire rods, their ten different sizes and colors.

According to the specifics of structured measuring tasks, in small groups, students interact with the Cuisenaire rods. Students first use informal and then formal fractional language to describe their actions and perceptions. For example, they may measure the length of a dark green rod with red rods and notice that its length equals the length of three red rods. Then, when they measure a red rod with a dark green rod, they will say that the length of a red rod equals one-third the length of a dark green rod. They will be introduced to the distinction between the measuring length or unit length and length to be measured. The students can notice that the length of a dark green rod equals three halves the length of a purple rod or six fourths the length of a purple rod. This noticing can lead to an awareness of equivalences.

Students will orally describe their observations and actions to each other, validating their work and the work of others. This work is done orally with the rods and represents the first of two nonsymbolic phases. After attaining facility with such rod arrangements and their corresponding spoken statements, in the second nonsymbolic phase, they will work without manipulating rods and create oral statements summarizing fractional relationships. These statements will be similar to the ones they made in the previous phase. In the next phase, students will be taught how to symbolize mathematically their statements. For instance, if a student stated, nine-thirds of 12 is bigger than ten-ninths of 18, the group of students will be shown that statement's symbolic representation: $\frac{9}{3} \times 12 > \frac{10}{9} \times 18$. Afterward, students will be invited to author statements such as $\frac{1}{2} \times \left(\frac{4}{5} \times 15 \right) = \frac{3}{4} \times \left(\frac{8}{3} \times 3 \right)$. Such statements, when authored by students based on relations among objects that they arrange or imagine,

provide evidence of their mathematical agency. Finally, students discuss and symbolize variants, invariants, and generalizations they have noticed. For example, they may have noticed that specific measures (fraction-*of*) of the same quantity are equivalent the same quantity such as these measures: $\frac{1}{2} \times, \frac{2}{4} \times, \frac{3}{6} \times, \frac{4}{8} \times$, and so on. This awareness and written generalization of it represents the fourth phase of the 4A-Instructional Model (see, POWELL, 2018b, for details).

As an example of the study of fraction-*as*-number, students learn to work with symbolic representations of fractions. For instance, they can be presented with pairs of symbolic fractions such as $^4/_9$ and $^3/_5$ and asked to determine their relative magnitudes. To decide, employing rod arrangements and reasoning developed during the initial sessions, they will construct a unit length and, based on the two given fractions, find other lengths that are appropriately proportional to it. As modeled in Figure 4, students will configure a line of four orange rods and one yellow rod (45) as the unit length and notice that $^3/_5$ of its length—the three blue rods (27)—is greater than the length of $^4/_9$ of the unit length—the four yellow rods (20). Students will learn to conceive of a fraction as a holistic quantity, representing a multiplicative comparison, rather than a componential entity of two natural numbers, its numerator and denominator.

Figure 4 – Comparing the magnitudes of two fractions, using a continuous model, Cuisenaire rods. The top line represents the chosen unit length, the middle line measures three-fifths of the unit length, and the bottom line measures four-ninths the length of the unit. This representation of relative magnitudes shows that the fraction whose value is four-ninths is less than the one that equals three-fifths.

FINAL CONSIDERATIONS

We believe that the proposed epistemology of fraction knowledge—a measuring perspective—offers several advantages. First, it relates fractions to its historical origins and, as such, restores its ontological roots. Second, our approach overcomes the documented conceptual difficulties of the traditional, dominant part/whole conception of fractions as the act of measuring challenges the current instructional sequence that places mixed numbers at the end of fraction learning and makes improper fractions proper consequences of multiplicative comparisons between pairs of quantities. Third, the

measuring approach helps to conceive a quotient of two natural numbers as a holistic magnitude. Fourth, the approach may mitigate the so-called whole or natural number basis. Moreover, our approach connects naturally with early-elementary activities around non-standard and standard measurement.

Our proposed view of fraction knowledge needs further theoretical development as well as empirical investigations. Theoretically, there are indications in the literature that provide substantiation for our proposal (BOBOS; SIERPINSKA, 2017; BROUSSEAU; BROUSSEAU; WARFIELD, 2004; DAVYDOV; TSVETKOVICH, 1991; DOUGHERTY; VENENCIANO, 2007; GATTEGNO, 1987; 1988; MORRIS, 2000; SCHMITTAU; MORRIS, 2004). Pedagogically, we draw on approach, called the subordination of teaching to learning (GATTEGNO, 1970d) and ideas about arithmetic learning on continous quantities rather than sets of discrete objects (CUISENAIRE; GATTEGNO, 1954; GATTEGNO, 1970a; b; c).

Empirically, we have studied teachers and students. From studying elementary pre-service teachers learning fractions as multiplicative comparisons—a measuring perspective—in a pre- and post-test study design, we found statistically significant changes in their ability to interpret fraction magnitudes using discrete and continuous models (ALQAHTANI; POWELL, 2018). We also observed that the pre-service teachers used the part/whole definition of fractions and its associated language to talk about comparing quantities multiplicatively. They would say "out of" and always calling the measuring rod "one" or "the whole," which prevented them from conceptualizing beyond fractions less than one.

To avoid these conceptual issues, we elected in a pilot investigation to work with second-grade students as they were without previous formal fraction instruction. As we designed our tasks for the second-graders, we paced the tasks accordingly and purposely structured them to invite the students to compare the lengths of the Cuisenaire rods. From this pilot investigation[3], we found that the second-grade students, working two hours per week for twelve weeks, are able to acquire and appropriate and articulate mathematical language to describe and record with mathematical notation perceived ratios between two lengths of Cuisenaire rods and respond flexibly and correctly in non-rehearsed situations

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[3] From 2018 to 2019, the pilot investigation was funded by a grant to the author and his doctoral student, Kendell V. Ali, from the Mathematics Education Trust of the National Council of Teachers of Mathematics.

of comparing multiplicatively two rods without the physical presence of the Cuisenaire rods (see Figure 5). Knowing how the rods' magnitudes compare to one another. to create these mathematical statements, the students evoked and manipulated their mental images of the Cuisenaire rods.

Figure 5 – Writing on a shared piece of chart paper, four different second-grade students authored these five fraction-of-quantity inequality statements: $\frac{1}{10} \times O < \frac{2}{10} \times O$, $\frac{3}{4} \times P < \frac{5}{7} \times E$, $\frac{3}{4} \times P = \frac{3}{4} \times Dg$, $\frac{4}{10} \times Q = \frac{2}{5} \times Q$, and $\frac{2}{4} \times P < \frac{3}{4} \times P$. Though the students did not have the Cuisenaire rods physically present, the letters refer to the color of the rods: O and Q = orange, P = purple, E = ebony, and Dg = dark green.

These students attended a poor-performing, urban school in an economically depressed community. Among the second-grade students of the school, the students in the pilot investigation were considered academically below average. Nevertheless, this pilot study evidences that a measuring perspective holds promise to address documented limitations of traditional approaches to fractions knowledge and broader societal issues engendered by these limitations. The limitations affect the broadening and diversity of participates in not only mathematics but also generally in science, technology, and engineering fields. The continued under-participation in higher mathematics of students like the ones in the pilot investigation has severe social and economic consequences especially for individuals from underrepresented communities.

REFERENCES

ALEKSANDROV, A. D. **A general view of mathematics.** In: ALEKSANDROV, A. D.; KOLMOGOROV, A. N.; LAVRENT'EV, M. A. (Org.). Mathematics: Its content, methods, and meaning. Cambridge, MA: Massachusetts Institute of Technology, 1963. v.1 p. 1-64.

ALQAHTANI, M. M.; POWELL, A. B. **Investigating how a measurement perspective changes pre-service teachers' interpretations of fractions.** In: HODGES, T. E.; ROY, G. J.; TYMINSKI, A. M. (Org.). Proceedings of the 40th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Greenville, SC: University of South Carolina & Clemson University., 2018. p. 484-487.

BAILEY, D. H. et al. Competence with fractions predicts gains in mathematics achievement. **Journal of Experimental Child Psychology**, v. 113, n. 3, p. 447-455, 11//. 2012. Disponível em: <http://www.sciencedirect.com/science/article/pii/S0022096512001063>.

BEHR, M. J. et al. **Rational number, ratio and proportion.** In: GROWS, D. A. (Org.). Handbook on research on mathematics teaching and learning. New York: Macmillan, 1992. p. 296-333.

BEHR, M. J. et al. **Rational number concepts.** In: LESH, R.; LANDAU, M. (Org.). Acquisition of mathematical concepts and processes. New York: Academic, 1983. p. 91-125.

BEHR, M. J. et al. Order and equivalence of rational numbers: A clinical teaching experiment. **Journal for Research in Mathematics Education**, v. 5, n. 5, p. 321-341, 1984.

BOBOS, G.; SIERPINSKA, A. Measurement approach to teaching fractions: A design experiment in a pre-service course for elementary teachers. **International Journal for Mathematics Teaching and Learning**, v. 18, n. 2, p. 203-239, 2017.

BOOTH, J. L.; NEWTON, K. J. Fractions: Could they really be the gatekeeper's doorman? **Contemporary Educational Psychology**, v. 37, n. 4, p. 247-253, 10//. 2012. Disponível em: <http://www.sciencedirect.com/science/article/pii/S0361476X12000392>.

BROUSSEAU, G.; BROUSSEAU, N.; WARFIELD, V. Rationals and decimals as required in the school curriculum: Part 1: Rationals as measurement. **The Journal of Mathematical Behavior**, v. 23, n. 1, p. 1-20, //. 2004. Disponível em: <http://www.sciencedirect.com/science/article/pii/S0732312303000683>.

CAMPOS, T. M. M.; RODRIGUES, W. R. A idéia de unidade na construção do conceito do número racional. REVEMAT - **Revista Eletrônica de Educação Matemática**, v. 2, n. 4, p. 68-93, 2007.

CARAÇA, B. D. J. **Conceitos fundamentais da Matemáti**ca. Lisboa: Tipografia Matemática, 1951.

CLAWSON, C. C. **The mathematical traveler:** Exploring the grand history of numbers. Cambrige, MA: Perseus, 1994/2003.

COURANT, R.; ROBBINS, H. **What is mathematics?:** An elementary approach to ideas and methods. 2nd revised by Ian Steward,. ed. Oxford: New York, 1941/1996.

CUISENAIRE, G.; GATTEGNO, C. **Numbers in colour:** A new method of teaching the process of arithmetic to all level of the Primary School. London: Hienemann, 1954.

DAVYDOV, V. V.; TSVETKOVICH, Z. H. On the objective origin of the concept of fractions. **Focus on Learning Problems in Mathematics**, v. 13, n. 1, p. 13-64, 1991.

DOUGHERTY, B. J.; VENENCIANO, L. C. H. Measure Up for Understanding. **Teaching Children Mathematics**, v. 13, n. 9, p. 452-456, 2007. Disponível em: <http://www.jstor.org.proxy.libraries.rutgers.edu/stable/41198995>.

DUFFY, S.; HUTTENLOCHER, J.; LEVINE, S. It is all relative: How young children encode extent. **Journal of Cognition and Development**, v. 6, n. 1, p. 51-63, 2005.

ESCOLANO VIZCARRA, R.; GAIRÍN SALLÁN, J. M. Modelos de medida para la enseñanza del número racional en Educación Primaria. Unión. **Revista Iberoamericana de Educación Matemática**, n. 1, p. 17-35, 2005.

FLAGG, G. **Numbers:** Their history and meaning. Mineola, NY: Dover, 1983.

GATTEGNO, C. **Gattegno mathematics text-book 1:** Qualitative arithmetic, The study of numbers from 1 to 20. New York: Educational Solutions, 1970a.

_____. **Gattegno mathematics text-book 2:** Study of numbers up to 1,000; The four operations. New York: Educational Solutions, 1970b.

_____. **Gattegno mathematics text-book 4:** Fractions, decimals, and percentages. New York: Educational Solutions, 1970c.

_____. **What we owe children:** The subordination of teaching to learning. New York: Avon, 1970d.

_____. **The science of education:** Part 1: Theoretical considerations. New York: Educational Solutions, 1987.

_____. **The science of education:** Part 2B: The awareness of mathematization. New York: Educational Solutions, 1988.

GATTEGNO, G. C.; HOFFMAN, M. R. **Handbook of activities for the teaching of mathematics at the elementary school.** New York: Human Education, 1976.

GILLINGS, R. J. **Mathematics in the time of the pharaohs.** Mineola, New York: Dover, 1972/1982.

GÓMEZ, D. M. et al. The effect of inhibitory control on general mathematics achievement and fraction comparison in middle school children. **ZDM Mathematics Education**, v. 47, n. 5, p. 801-811, 2015. Disponível em: <https://doi.org/10.1007/s11858-015-0685-4>.

ISCHEBECK, A. K.; SCHOCKE, M.; DELAZER, M. The processing and representation of fractions within the brain. An fMRI investigation. **NeuroImage**, v. 47, p. 403-413, 2009/07/01. 2009. Disponível em: <http://www.sciencedirect.com/science/article/pii/S1053811909706937>.

JACOB, S. N.; NIEDER, A. Notation-Independent Representation of Fractions in the Human Parietal Cortex. **Journal of Neuroscience**, v. 29, n. 14, p. 4652-4657, 2009a. Disponível em: <http://www.jneurosci.org/content/jneuro/29/14/4652.full.pdf>.

_____. Tuning to non-symbolic proportions in the human frontoparietal cortex. **European Journal of Neuroscience**, v. 30, n. 7, p. 1432-1442, 2009b. Disponível em: <https://onlinelibrary.wiley.com/doi/abs/10.1111/j.1460-9568.2009.06932.x>.

KERSLAKE, D. **Fractions:** Children's strategies and errors. Windsor, England: NEER-NELSON, 1986.

KIEREN, T. E. **On the mathematical, cognitive and instructional foundations of rational number.** In: LESH, R. A. (Org.). Number and measurement. Columbus, Ohio: ERIC Clearinghouse for Science, Mathematics, and Environmental Education, 1976. p. 101-144.

_____. **Personal knowledge of rational numbers:** Its intuitive and formal development. In: HEIBERT, J.; BEHR , M. J. (Org.). Number concepts and operations in the middle grades. Reston, VA: Lawrence Erlbaum, 1988. p. 49-84.

LAMON, S. J. **Presenting and representing:** From fractions to rational numbers. In: CUOCO, A.; CURCIO, F. (Org.). The Roles of Representations in School Mathematics - 2001 Yearbook. Reston: National Council of Teachers of Mathematics, 2001. p. 146-168.

_____. **Rational numbers and proportional reasoning:** Toward a theoretical framework for research. In: LESTER, J. F. K. (Org.). Second handbook of research on mathematics teaching and learning: A Project of the National Council of Teachers of Mathematics. Charlotte, NC: Information Age, 2007. p. 629-667.

_____. **Teaching fractions and ratios for understanding:** Essential content knowledge and instructional strategies for teachers. 3rd. ed. New York: Routledge, 2012.

LEWIS, M. R.; MATTHEWS, P. M.; HUBBARD, E. M. Neurocognitive architectures and the nonsymbolic foundations of fractions understanding. In: BERCH, D. B.; GEARY,

D. C.; KOEPKE, K. M. (Org.). **Development of mathematical cognition:** Neural substrates and genetic influences. San Diego, CA: Academic Press., 2015. p. 141–160.

LIN, C.-Y. et al. Preservice Teachers' Conceptual and Procedural Knowledge of Fraction Operations: A Comparative Study of the United States and Taiwan. **School Science and Mathematics**, v. 113, n. 1, p. 41-51, 01/01/. 2013.

MAHER, C. A.; YANKELEWITZ, D., Eds. **Children's reasoning while building fraction ideas.** Boston: Senseed. 2017.

MATTHEWS, P. G.; CHESNEY, D. L. Fractions as percepts? Exploring cross-format distance effects for fractional magnitudes. **Cognitive Psychology**, v. 78, p. 28-56, 2015.

MATTHEWS, P. G.; ELLIS, A. B. **Natural alternatives to natural number:** The case of ratio. Journal of Numerical Cognition, v. 4, n. 1, p. 19-58, 2018.

MATTHEWS, P. G.; ZIOLS, R. **What's perception got to do with it? Re-framing foundations for rational number concepts.** In: NORTON, A.; ALIBALI, M. W. (Org.). Constructing Number: Merging Perspectives from Psychology and Mathematics Education. Cham: Springer International Publishing, 2019. p. 213-235.

MCCRINK, K.; WYNN, K. Ratio abstraction by 6-month-old infants. **Psychological Science**, v. 18, n. 8, p. 740-745, 2007.

MOCK, J. et al. Magnitude processing of symbolic and non-symbolic proportions: An fMRI study. **Behavioral Brain Functions**, v. 14, n. 9, p. 1-19, 2018.

MORRIS, A. K. A teaching experiment: Introducing fourth graders to fractions from the viewpoint of measuring quantities using Davydov's mathematics curriculum. **Focus on Learning Problems in Mathematics**, v. 22, n. 2, p. 32-83, 2000.

NATIONAL MATHEMATICS ADVISORY PANEL. **Foundations for success:** The final report of the National Mathematics Advisory Panel. Washington, DC: US Department of Education, 2008.

NI, Y.; ZHOU, Y.-D. Teaching and learning fraction and rational numbers: The origins and implications of whole number bias. **Educational Psychologist**, v. 40, n. 1, p. 27-52, 2005.

OBERSTEINER, A. et al. **Understanding fractions:** Integrating results from mathematics education, cognitive psychology, and neuroscience. In: NORTON, A.; ALIBALI, M. W. (Org.). Constructing number: Merging perspectives from psychology and mathematics Education. Cham: Springer, 2019. p. 135-162.

OECD. **A profile of student performance in mathematics.** In: PISA 2012 results: What students know and can do—Student performance in mathematics, reading, and science. Paris: OECD Publishing, 2014. v.1, Revised edition, February 2014 p. 31-144.

POWELL, A. B. Melhorando a epistemologia de números fracionários: Uma ontologia baseada na história e neurociência. **Revista de Matemática, Ensino e Cultura**, v. 13, n. 29, p. 78-93, 2018a.

_____. Reaching back to advance: Towards a 21st-century approach to fraction knowledge with the 4A-Instructional Model. **Revista Perspectiva**, v. 36, n. 2, p. 399-420, 2018b.

POWELL, A. B.; ALI, K. V. **Design research in mathematics education:** Investigating a measuring approach to fraction sense. In: CUSTÓDIO, J. F. et al. (Org.). Programa de Pós-Graduação em Educação Científica e Tecnológica (PPGECT): Contribuições para pesquisa e ensino. São Paulo: Livraria da Física, 2018. p. 221-242.

RESNIKOFF, H. L.; WELLS JR., R. O. **Mathematics in civilization.** New York: Dover, 1973/1984.

RITCHIE, S. J.; BATES, T. C. Enduring links from childhood mathematics and reading achievement to adult socioeconomic status. **Psychological Science**, v. 24, n. 7, p. 1301-1308, 2013.

ROQUE, T. **História da matemática:** Uma visão crítica, desfazendo mitos e lendas. Rio de Janeiro: Zahar, 2012.

SCHEFFER, N. F.; POWELL, A. B. Frações nos livros do programa nacional do livro didático brasileiro (PNLD). **Revemop**, in press.

SCHMITTAU, J.; MORRIS, A. The development of algebra in the elementary mathematics curriculum of V.V. Davydov. **The Mathematics Educator**, v. 8, n. 1, p. 60-87, 2004.

SIEGLER, R. S. Magnitude knowledge: The common core of numerical development. **Developmental Science**, v. 19, n. 3, p. 341-361, 2016.

SIEGLER, R. S. et al. Early Predictors of High School Mathematics Achievement. **Psychological Science**, v. 23, n. 7, p. 691-697, 2012.

SIEGLER, R. S. et al. Fractions: the new frontier for theories of numerical development. **Trends in Cognitive Sciences**, v. 17, n. 1, p. 13-19, 1//. 2013. Disponível em: <http://www.sciencedirect.com/science/article/pii/S1364661312002653>.

SIEGLER, R. S.; LORTIE-FORGUES, H. An integrative theory of numerical development. **Child Development Perspective**, v. 8, n. 3, p. 144-150, 2014.

SON, J.-W.; SENK, S. L. How reform curricula in the USA and Korea present multiplication and division of fractions. **Educational Studies in Mathematics**, v. 74, n. 2, p. 117-142, 2010. Disponível em: <http://www.jstor.org/stable/40603199>.

SOPHIAN, C. Perceptions of proportionality in young children: Matching spatial ratios. **Cognition**, v. 75, n. 2, p. 145-170, 2000.

STRUIK, D. J. **A concise history of mathematics.** 3rd Revised. ed. New York: Dover, 1948/1967.

SWELLER, J. Cognitive load theory, learning difficulty, and instructional design. **Learning and instruction**, v. 4, n. 4, p. 295-312, 1994.

SWELLER, J.; VAN MERRIENBOER, J. J.; PAAS, F. G. Cognitive architecture and instructional design. **Educational psychology review**, v. 10, n. 3, p. 251-296, 1998.

TORBEYNS, J. et al. Bridging the gap: Fraction understanding is central to mathematics achievement in students from three different continents. **Learning and Instruction**, v. 37, p. 5-13, 6//. 2015. Disponível em: <http://www.sciencedirect.com/science/article/pii/S0959475214000255>.

TUCKER, A. **Fractions and units in everyday life.** In: MADISON, B. L.; STEEN, L. A. (Org.). Calculation vs. context: Quantitative literacy and its implications for teacher education. Washington, DC: Mathematical Association of America, 2008. p. 75-86.

TZUR, R. An integrated study of children's construction of improper fractions and the teacher's role in promoting that learning. **Journal for Research in Mathematics Education**, v. 30, n. 4, p. 390-416, 1999.

VAMVAKOUSSI, X.; VAN DOOREN, W.; VERSCHAFFEL, L. Naturally biased? In search for reaction time evidence for a natural number bias in adults. **The Journal of Mathematical Behavior**, v. 31, n. 3, p. 344-355, 2012/09/01/. 2012. Disponível em: <http://www.sciencedirect.com/science/article/pii/S0732312312000144>.

WU, H.-H. How to prepare students for algebra. **American Educator**, v. 25, n. 2, p. 1-7, 2001.

_____. What's sophisticated about elementary mathematics? Plenty—That's why elementary schools need math teachers. **American Educator**, v. 33, n. 3, p. 4-14, 2009.