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REVISITING THE ARMS RACE RICHARDSON'S MODEL: BEYOND THE TWO ACTORS' APPROACH

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Resumo

O modelo de Richardson é clássico no campo das Relações Internacionais, ao descrever a corrida armamentista entre dois atores, por meio de sistema de equações diferenciais ordinárias com coeficientes constantes acerca dos respectivos orçamentos militares. Neste artigo, estendemos o modelo de Richardson para abarcar um número arbitrário de atores e investigar se existem fatores de escala que surgem quando vários atores são considerados, em primeiro lugar, tratando o caso especial quando os atores são indiferenciados e, em segundo lugar, indagamos sobre o caso de atores diferenciados. Relatamos que, à medida que aumenta o número de atores, não há garantia de que os orçamentos individuais não tendam a aumentar sem limites, o que é um resultado teórico não presente no modelo original e apresenta novas possibilidades de se pensar os limites da estabilidade do sistema internacional.

Palavras-chave: Relações Internacionais, modelo de Richardson, equações diferenciais ordinárias.

Abstract

Richardson's model is classical in the field of International Relations, which describes arms race between two actors, by means of system of ordinary differential equations with constant coefficients of the respective military budgets. In this paper, we extend Richardson's model to comprise an arbitrary number of actors to investigate whether there are scale properties that arise when multiple actors are considered, firstly by treating the special case when the actors are undifferentiated, then the case of differentiated actors. We report that, as the number of actors increases, there is no guarantee that individual budgets will not tend to increase limitlessly, which is a theoretical result not present in the original model and presents new possibilities of thinking about the boundaries of international system stability.

Keywords: International Relations, Richardson's model, ordinary differential equations.

1 Introduction

Few historical events have changed the international landscape as profoundly and as unexpectedly as the end of the Cold War. The sudden implosion of the Soviet Union, one of the poles of power that dictated the international order since 1945, and challenged the capitalist order since 1922, presented new possibilities and expectations for the international system, such as the possibility of allocating the budget previously applied in defence to foster development, the so-called "Peace dividends", and the prospect of full functioning of the main international institutions, halted due to the American-Soviet dispute, one of the most illustrative cases being the activation of the collective security mechanism by United Nations Security Council on the onset of the First Gulf War (1990). Such events at the phenomenological level have not ceased to be worked on by the Academy, as we observe the renewal of the neoliberal-neorealist debate (or simply neo-neo debate) in the International Relations (IR) field, with emphasis on the post-1991 situation for the construction of the new international order, that is , "after (U.S.) victory" [11]. However, entering the new millennium, expectations were not entirely confirmed, and we observed disregard for international institutions and International Law, in episodes as dramatic as the Second Gulf War (2003), annexation of Crimea (2014) and the continued construction of artificial islands in the South China Sea. More than that, the period after the expected Fukuyaman "end of history" [9] not only did not confirm the expectations for military spending reduction, but also witnessed persistent growth among the main military great and middle powers [24]. Empirical phenomena instigated theoretical developments in the neo-neo debate, with the advance of offensive neorealism [16] and reconsideration of the initial neoliberal position [12, p. 6]. A renewed interest about the behaviour of countries in their military spending was sparked in the IR field.

In this context, we turn to the Richardson's model, one of the most studied formal approaches in the IR discipline, with wide use and adaptation in specific cases such as "the military expenditure of France and Russia and of Germany and Austria in the period between 1909 and 1914 " [8, p. 293], and the military expenditures of the North Atlantic Treaty Organisation and the Warsaw Pact during the Cold War [7]. The model describes how rational actors control defence spending in response to the behaviour of other actors, with special attention to interactive trends between actors (arms race). The contribution of this work to the literature is the expansion of scope of the Richardson's model to accommodate the case of several actors, in comparison to the 2 actors case extensively applied in the literature, endowing the expanded model with greater analytical sophistication, as observed by [26, p. 27], and the consideration of systemic factors to the arms race, especially in regard to the effects of the system scale to individual actors. We report that no international system is able to maintain stability for an arbitrary number of actors.

The paper is organised in 5 parts. First, we frame our analysis in the historical and theoretical debate in IR. Second, we describe the classical Richardson's model for two actors. Third, we present the multivariate case and discuss structural implications for the special case of multiple undifferentiated actors. Fourth, we extend some of the main findings of the previous session to multiple differentiated actors. Finally, we conclude, addressing some of the limitations of the model and other considerations.

2 Theoretical debate

After the *interregnum* of the immediate post-cold war, when the structuralist IR theories of neoliberal and neorealist suffered a major crisis, either by the non-confirmation of their conclusions with the unexpected and predominantly domestic end of the Soviet Union, or by new the post-positivist approaches, both theoretical currents returned to the central *locus* in IR theorising. Not only did reference works emerge within each current with new ideas, such as Ikenberry's *After Victory* [11] and Mearsheimer's *Tragedy of Great Power Politics* [16], but there was also a *real* debate among the main authors of each current in periodicals, especially in the *International Organization* journal, and other compiled books.

The neo-neo debate is extensive, covering a myriad of authors and controversial topics, which [3, p. 4-8] summarises in six points focal points - nature and consequences of anarchy, international cooperation, relative gains and absolute gains, priority of state objectives, intentions and capacities and, finally, institutions and regimes. More fruitful than trying to reconstruct the step-by-step of this broad discussion, we find it more useful for the purposes of this paper to synthetically present key ideas of the two structuralist currents, considered as ideal types, under a common analytical framework that allows them to establish theoretical approximations relevant to our endeavour to understand the behaviour of actors regarding military spending. The common ground of positivist premises and rationalist approach by both neorealists and neoliberals is fundamental to justify the choice of the analytical framework of cooperative game theory to schematise our section of the neo-neo debate.

Neoliberalism recognises the rationality and self-interest of the international actors as neorealism [10, p.156] presumes them to be, but does not preclude that, even in an anarchic world, considered as the main cause of conflict from a neorealistic perspective, international cooperation is impossible. Thus, special emphasis should be placed on institutions and regimes, some of the main arrangements that are able to provide security as a public good, rather than self-help. Under such assumption, it is a favourable situation for the actors to engage in cooperative schemes and to allocate resources not only for defence, which remains relevant, as we will see, but mainly to economic gains in international interactions with other actors, even if it generates relative gains among actors. The concern with aggression or with asymmetries arising from differential potential of gains is minimised with the establishment of rules, compatible with individual interests and capable of promoting reciprocity between the actors [2, p. 110-111]. We should bear in mind that the prospect of non-aggression between two or more actors is a prerequisite for cooperation between them.

In this sense, the $i \in A = \{1, \ldots, n\}$ actors in the system are expected to engage in the large collective security coalition $C = \{1, \ldots, n\}$, in which no gain is profited from aggression against participating members $v(C) = 0$, that is, the allocation of resources to individual payoffs, strictly from a security perspective, is $\mathbf{x} = \mathbf{0}$. If a member j decides to take advantage by subjugating another member k with fewer capacities $m(k) < m(j)$, the gains $v(\{j\}) > 0$ are in principle superior to those resulting from their participation in the coalition C and security, as a public good, is threatened. However, for the regime to be fully operational, the other actors, or a subset of them, from $A - \{j\}$ must have the means to impose losses on the aggressor, therefore defence expending is not to be dismissed completely, so that any deviant country has no incentives to abandon the grand coalition, the case being generalisable also for deviant sub-coalitions. Since participation in the large coalition has the largest payoffs possible, it is in the interest of the actors to participate in this scheme in order to obtain maximum benefits.

Now, there is nothing to prevent a group of countries sufficiently endowed with capacities $S \subset A$, more than half of all available capabilities $\frac{m(S)}{m(A)} > 0.5$, to try to subdue the remaining actors of the system S^c. Therefore, $v(S) > v(C)$ and the game core [22, p. 239] is empty, that is, there is no resource allocation capable of satisfying all the possibilities of deviant coalitions. In fact, theorising about the difficulties of guaranteeing the integrity of the actors through cooperative means is treated by the neorealist school as constrained by systemic structures, determined by the distribution of material capabilities, the governing principle of the international system [25, p. 81]. Under this perspective, institutions are a mere reflection of that distribution and rules do not bind actors to expected behaviours. Our framework privileged the lack of a cooperative game balance solution as a complicating factor for the full functioning of institutions, but there are other dimensions that institutional arrangements can provide.

According to some strands of neoliberal thinking, "institutions can provide information, reduce transaction costs, make commitments more credible, establish focal points for coordination, and in general facilitate the operation of reciprocity" [14, p. 42]. Through these mechanisms, the actors modify their behaviour in favour of cooperative strategies, since they have greater transparency and have incentives to make decision considering the long term. However, there is no guarantee of symmetric or perfection information available in the cooperative game for all actors. Also, the shadow of the future may not be enough to change the behaviour of the actors, as a mistaken bet on the efficacy of the institutions may imply the end of the actor's future, so that the immediate present is more privileged than the long horizon, hence we should expect no complete trust of the credibility of the commitments made by members of the coalition. Thus, to neorealism, some of the main means that institutions have to change the behaviour of the actors have no or little effectiveness in this specific case [15, p. 19].

We recognise that the capacity of institutions and regimes to provide security can be conceived as a situation of local sub-equilibrium *lato sensu*, possibly semi-stable and not necessarily global. It remains for the neoliberal school to show us, *by formal methods*, what are the factors that, added to the previous thinking, effectively make States participate in the coalition or other institutional arrangements. On the one hand, it is a more than welcome contribution to the debate, as they are phenomena that are observable in international nature, but are not considered by neorealism [21, p. 30-31]. On the other hand, we focus on this uncertainty in relying on the ability of cooperative coalitions and institutions to function, a situation in which individual actors, rational and risk-averse, resort to self-help, with their own armaments, to face security challenges. Such behaviour is described by Richardson's model.

3 Preliminaries

The classical model [27, p. 180-189], originally by [23], is described by the following system of ordinary differential equations (ODEs),

$$\begin{cases} x'(t) = -b \cdot x(t) + a \cdot y(t) + c \\ y'(t) = -e \cdot y(t) + d \cdot x(t) + f. \end{cases} \tag{3.1}$$

We denote by $\mathbb{R}$ the set of real numbers and the set of non-negative real numbers as $\mathbb{R}^+$, and the set of positive real numbers as $\mathbb{R}^+_*$. The model 3.1 is described by $x, y : \mathcal{T} \to \mathbb{R}^+$, $\mathcal{T} = \{t \in \mathbb{R}^+, \text{such that}, (x(t), y(t)) \in \mathbb{R}^+_* \times \mathbb{R}^+_*\}$, $a, b, d, e \in \mathbb{R}^+_*$ and $c, f \in \mathbb{R}$ in the (3.1) system, or, more synthetically, $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}$, with $\mathbf{A} = \begin{vmatrix} -b & a \\ d & -e \end{vmatrix}$ and $\mathbf{b} = (c, f)$.

Here, $x(t)$ represents the military expenditure of a given country A at the time t, $y(t)$ being the situation corresponding to the rival country B. The variation of spending in country A, given by $x'(t)$, or simply $\dot{x}$, is directly proportional to the expenses incurred by country B $(a \cdot y)$, as no country would like to fall behind their competitors, but negatively proportional to the expenses already incurred $(-b \cdot x)$. The c factor is reserved to explain exogenous factors to military budgetary trends, from causes arising beyond the dyadic dynamics of expenditures, such as the advent of a period of economic prosperity in country A, which allows a greater budget for defence, regardless of the expenses incurred by country B. The initial value $x(0)$ is the expenditure at time $t = 0$, the beginning of the interaction between actors.

Rather than unimportant proportionality constants that regulate $x(t)$, one possible interpretation for the constants can be achieved through dimensional analysis. $\dot{x}$ represents the rate of change of budget spending over time and therefore has dimensions $\frac{currency}{time}$. Since the dimensions on the left side of the equation have to be equal to those on the right side, it follows that c also expresses the "speed" at which exogenous factors influence $x(t)$. Different is the nature of a and b. Since they already follow the investment, expressed in value dimensions (currency), they necessarily have the dimension $\frac{1}{time}$, also defined as frequency. Thus, they represent the rate at which the $\dot{x}$ variation takes place depending on the present budgets. The developments in these last two paragraphs is analogous to the $\dot{y}$ case.

The system (3.1) has as solution, for $\mathbf{A}$ diagonalisable [6, p. 339],

$$\mathbf{x}(t) = \mathbf{\Psi}(t)\mathbf{\Psi}^{-1}(0)\mathbf{x}(0) + \mathbf{\Psi}(t)\int_0^t \mathbf{\Psi}^{-1}(s)\mathbf{b}\ ds. \tag{3.2}$$

With corresponding fundamental matrix $\mathbf{\Psi}(t)$, for eigenvalues $\lambda_1 = \frac{-b-e-\sqrt{b^2+4ad-2be+e^2}}{2}$ and $\lambda_2 = \frac{-b-e+\sqrt{b^2+4ad-2be+e^2}}{2}$,

$$\mathbf{\Psi}(t) = \begin{pmatrix} \frac{(e+2\lambda_1)\mathbb{e}^{\lambda_1 t}}{2d} & \frac{(e+2\lambda_2)\mathbb{e}^{\lambda_2 t}}{2d} \\ \mathbb{e}^{\lambda_1 t} & \mathbb{e}^{\lambda_2 t} \end{pmatrix}.$$

More interesting than the mathematical developments that lead to the (3.2) solution, including existence and its uniqueness [13, p. 25], are the implications that arise from it, model-wise. Our first and possibly main issue of concern is whether the interaction between actors will lead to an unlimited arms race between them, that is, the long term behaviour of the system $\mathbf{x}(t)$, as t increases. To this end, we should inquire the eigenvalues' real part sign. Since $b > 0$ and $e > 0$, $\lambda_1 < 0$. For λ_2, we should first notice that $\det(\mathbf{A}) = be - ad$, therefore λ_2 can be written as $\frac{tr(\mathbf{A})}{2} + \sqrt{(\frac{tr(\mathbf{A})}{2})^2 - \det(\mathbf{A})}$. From our assumption that $\mathbf{A}$ is diagonalisable, $\det(\mathbf{A}) < 0$ implies $\lambda_2 > 0$ and $\det(\mathbf{A}) > 0$ implies $\lambda_2 < 0$, considering that $\lambda_2 \in \mathbb{R}$, since the term inside the square root $b^2 + 4ad - 2be + e^2$ is strictly non-negative, because $(b-e)^2 \geq 0$ and the product $ad > 0$, resulting $b^2 + 4ad - 2be + e^2 > 0$. Either both eigenvalues are negative, and the system is stable, or one of them is positive, and the system is unstable, unless, for this last case, $\mathbf{x}(0)$ is equal to the critical point, which is unlikely to happen, should we take an arbitrary starting point.

The key role presented by $\det(\mathbf{A})$ sign in deciding λ_2 sign and the overall system behaviour as $\det(\mathbf{A})$ may be interpreted as a measure between the stabilising and destabilising trends of actors' expenditure. The geometric meaning of $det(\mathbf{A})$ in $\mathbb{R} \times \mathbb{R}$ is the parallelogram area formed by vectors $\mathbf{v}_1 = (-b, a)$ and $\mathbf{v}_2 = (d, -e)$, being $-b$ and $-e$ the balancing factors, as they promote decreased military budgets, against a and d with inverse effects. Then,

$$\begin{aligned}\det(\mathbf{A}) &= be - ad \\ &= ||v_1|| \cos(\theta_1)||v_2|| \sin(\theta_2) - ||v_1|| \sin(\theta_1)||v_2|| \cos(\theta_2) \\ &= ||v_1||||v_2|| \sin(\theta_2 - \theta_1), \text{ for } \theta_1 \in [\frac{\pi}{2}, \pi] \text{ and } \theta_2 \in [\frac{3\pi}{2}, 2\pi]\end{aligned}$$

Whenever there is a relative angular convergence greater than π, $\theta_2 - \theta_1 < \pi$, $\det(\mathbf{A}) < 0$, which is a curious observation. Unless we are treating an extreme case ($\theta_1 = \pi$ or $\theta_2 = 2\pi$), not only the overall system behaviour is dependent on inputs from both actors, represented by a relative coefficient, as would be expected since each actor's behaviour influence the other, but it is also possible to always find θ_1 for fixed θ_2, and vice-versa, that satisfies the above inequality to generate negative eigenvalues.

The stability analysis also pervades the notion of a critical point in the system. The critical point is a fixed point in the space of values that the system can assume, in which $\mathbf{x}(t)$ remains constant, regardless of the systemic temporal evolution. It corresponds to the situation when individual actors have no interest in altering military budgets. In addition, the critical point is an important reference to know if other initial values of the $\mathbf{x}(0)$ system tend to approach or move away from stability. The critical point $\mathbf{x}_c$, which satisfies $\mathbf{0} = \mathbf{A}\mathbf{x}_c + \mathbf{b}$, is retrieved by Cramer's rule, yielding

$$\mathbf{x}_c = \left(\frac{\begin{vmatrix} c & a \\ f & -e \end{vmatrix}}{\begin{vmatrix} -b & a \\ d & -e \end{vmatrix}}, \frac{\begin{vmatrix} -b & c \\ d & f \end{vmatrix}}{\begin{vmatrix} -b & a \\ d & -e \end{vmatrix}} \right). \tag{3.3}$$

As per $\det(\mathbf{A})$ case, $\mathbf{x}_c$ also represents a measure between stabilising and destabilising trends in the system, taking into account the exogenous factors c and f relative to the overall $\det(\mathbf{A})$ systemic behaviour. Should there not be exogenous factors, $\mathbf{x}_c = \mathbf{0}$ and either the individual expenditures reach 0 or deviate from 0.

The case for non-diagonalisable $\mathbf{A}$, i.e. $\det(\mathbf{A}) = 0$, is slightly different, as $\lambda_1 < 0$ and $\lambda_2 = 0$, still holding some kind of stability. However, one should not think of $\mathbf{x}_c$ as a point, but rather a line, since $\det(\mathbf{A}) = 0$ implies that the matrix $\mathbf{A}$ does not have row vectors linearly independent and $\exists k \in \mathbb{R} \setminus \{0\}$, such that $(-b, a) = k(d, -e)$, and for (3.3), $\mathbf{x}_c \in \{(x_c, y_c), \text{such that } -bx_c + ay_c = \frac{c+kf}{2}\}$.

At first glance, system 3.1 captures everything we need, in order to describe the dynamics of military expenditure between two countries. However, it does not seem to be true for the general case. Depending on initial point and coefficient values, for sufficiently large $t = \tau$, we may find $x(\tau)$ or $y(\tau)$ reaching 0 or negative values, which does not make sense model-wise, on the one hand because of the physical impossibility of negative investments, on the other hand, the possibility of a country achieving decreasing levels of material capabilities, until it reaches 0. In this sense, we recognise that the domain of the constants is not as free as we have firstly considered, in order for our model not to produce degenerate cases. Despite being the simplest system, composed of only two actors, at the current level of our research, we are still unable to relate the constants $(a, b, c, d, e, f, g, x_0, y_0)$ in a meaningful fashion, other than conditions under which $x(t) > 0$ and $y(t) > 0$, for $\forall t \in \mathcal{T}$. Therefore, eigenvalue analysis is necessary but insufficient to inquire about the asymptotic behaviour for the system 3.1. Such gap should be treated in future works, but for the sake of the multivariate investigation, our main focus in the present article, we will show the appropriate domain of the respective constants for both the undifferentiated and differentiated actors' cases.

As interesting and explanatory the two actors' model is, it may not be comprehensible enough to account for the structural effects possibly arising from a system with an arbitrary number of actors. To investigate whether such effects indeed do exist, we extend the (3.1) model to include $n \in \mathbb{N}, n > 2$ actors, with the same observations above about the constants' domain, meanings of the variables being valid and $\mathcal{T} = \{t \in \mathbb{R}_*^+, \text{such that}, x_i(t) \in \mathbb{R}_*^+\}$, $\mathbf{x}' = \mathbf{M} \cdot \mathbf{x} + \mathbf{c}$, the ODE system is described as, with coefficients $a_{ij} \in \mathbb{R}_*^+$, $i, j \in \{1, \ldots, n\}$, and $b_i \in \mathbb{R}$, $i \in \{1, \ldots, n\}$,

$$\begin{cases} \dot{x}_1 = -a_{11} \cdot x_1 + \sum_{i=2}^{n} a_{1i} \cdot x_i + b_1 \\ \vdots \\ \dot{x_j} = -a_{jj} x_j + \sum_{i \in \{1,\ldots,n\}, i \neq j}^{n} a_{ji} x_i + b_j \\ \vdots \\ \dot{x}_n = -a_{n1} \cdot x_n + \sum_{i=1}^{n-1} a_{ni} \cdot x_i + b_n. \end{cases} \tag{3.4}$$

4 Special case: undifferentiated actors

The model (3.4) is certainly a more complex system, whose analytical solution can be an arduous task in the case of indeterminate coefficients, since for $n > 5$ it is not always possible to solve the algebraic equations resulting from the search for eigenvalues, since there are no general solutions for equations of quintic degree or higher. Thus, we consider the non-homogeneous case for the $n \geq 2$ dimensional squared $\mathbf{M} = (a_{ij})$ matrix, whose indexed elements are $a_{ii} = -b, b \in \mathbb{R}_*^+$, $a_{ij} = a \in \mathbb{R}_*^+$, and corresponding constant vector $\mathbf{b}$, with entries $b_i = c \in \mathbb{R}, \forall i \neq j \leq n$, which means that they are the same for all actors, thus, constituting an international system of similarly behaving actors.

It is trivial to show that the general solution for the i-th actor is, extending (3.2) and bearing in mind that there is one $\lambda_1 = -b + (n-1) \cdot a$ eigenvalue, corresponding to eigenvector $\epsilon_1 = (1, \dots, 1) = \mathbf{1}$ and $n-1$ repeated eigenvalues $\lambda_{i \neq 1} = -a - b$ for $\mathbf{M}$, with corresponding eigenvectors $\epsilon_2 = (1, 0, \dots, -1)$, $\epsilon_3 = (0, 1, 0, \dots, -1)$, $\dots$, $\epsilon_n = (0, 0, \dots, 1, -1)$,

$$x_i(t) = \mathbb{e}^{(-b+(n-1)\cdot a)\cdot t} \cdot \left(\frac{\sum_j^n x_j(0)}{n} - \frac{c}{b-(n-1)\cdot a} \right) + \left(x_i(0) - \frac{\sum_j^n x_j(0)}{n} \right) \cdot \mathbb{e}^{(-b-a)\cdot t} + \frac{c}{b-(n-1)\cdot a}. \quad (4.1)$$

We know that the relative proportion between domestic and external trends is fundamental to determine individual behaviours and that, in addition, initial investments only influence the pace of military spending, which, once again, lead us back to eigenvalue signal analysis. Knowing that the stability of the system is dependent on the eigenvalue $-b + (n-1) \cdot a$, as more actors participate in the system, stability is threatened, since $\lim_{n \to \infty} \lambda_1 = +\infty > 0$, and the corresponding eigenvector ϵ_1 is the only attractor to all valid point in $(\mathbb{R}_*^+)^n = \mathbb{R}_*^+ \times \cdots \times \mathbb{R}_*^+$, the n-times Cartesian product of $\mathbb{R}_*^+$.

Ultimately, it is possible to conclude that the system cannot support an arbitrary number of actors, regardless of the domestic, external and exogenous trends of the model, and remain stable at the same time, with a narrowing window of stability, illustrated in Figure 1. In it, we represent the superposed domains in $\mathbb{R}_*^+ \times \mathbb{R}_*^+$, for

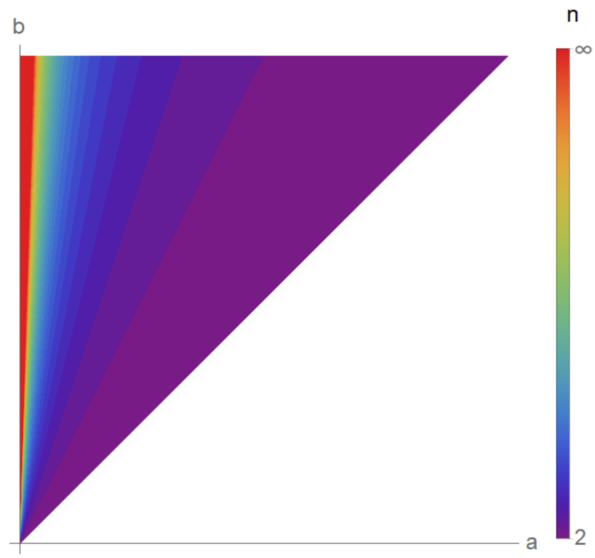

Figure 1: Negative eigenvalues' domain

which we may find negative eigenvalues, coloured by hue (colour tone) according to the number of actors in the system, from the minimal number of two actors (purple), occupying the undermost plane, to an arbitrary large number of actors (red) on the uppermost plane. As the number of actors increases, the domain of stability narrows, covering diminishing area from the Cartesian plane, until collapsing on the b axis, where a equals zero, and corresponds to the degenerate case of an international system composed of actors without incentives to pursue greater defence expending, other than due to exogenous factors, irrespective of military capabilities differences among them.

Stability is one of our main interests, but there are other systemic variables that deserve analytical treatment, such as the critical point or how inequalities in initial investment evolve over time. As the critical point is fixed, we take $\mathbf{x}'(t) = \mathbf{M} \cdot \mathbf{x}(t) + \mathbf{c}$ for a time when there is no variation in individual expenses, that is, $-\mathbf{c} = \mathbf{M} \cdot \mathbf{x}$, where a linear system is formed, whose solution, easily found, is $\mathbf{x}_c = (\frac{-c}{-b+(n-1)\cdot a}, \ldots, \frac{-c}{-b+(n-1)\cdot a})$. More than a simple algebraic expression, $\frac{-c}{-b+(n-1)\cdot a}$ represents the ratio between factors exogenous to the model (c) and the trends arising from the system, both domestic ($-b$), and international ($(n-1) \cdot a$). As, $n \to \infty$, $\mathbf{x}_c \to \mathbf{0}$, which means that, regardless of different constant values, provided that the respective domains are respected, the critical point will move towards the origin and the system will deviate from the origin point, when $t \to \infty$. Combined with our knowledge that negative eigenvalues are harder to find as the number of actors increases, the system will deviate from such critical point, as it constitutes a source (opposed to a sink or a saddle point) of the ODEs' equilibrium point.

Investigating systemic inequality, an adequate indicator of analysis is the variance of the values $x_i(t)$ between the actors, from solution (4.1). First, we need to average $x_i(t)$,

$$\frac{\sum_i^n x_i(t)}{n} = \mu(t) = \mathbb{e}^{(-b+(n-1)\cdot a)t} \cdot \left(\frac{\sum_j^n x_j(0)}{n} - \frac{c}{b-(n-1)\cdot a} \right) + \frac{c}{b-(n-1)\cdot a}.$$

The behaviour of the average is similar to that of individual actors. According to the eigenvalue λ_1, the average either grows continuously or tends to $\frac{c}{b-(n-1)\cdot a}$, which corresponds to situations of instability and stability, respectively, except for the singular case when the average of the initial investments is equal to $\frac{c}{b-(n-1)\cdot a}$.

With the average, it is possible to calculate the variance,

$$\sigma^2(t) = \frac{\sum_i^n \left(x_i(t) - \frac{\sum_j^n x_j(t)}{n} \right)^2}{n} \Rightarrow \sigma^2(t) = \frac{\sum_i^n (x_i(0) - \mu(0))^2 \cdot \mathbb{e}^{(-ba)\cdot 2t}}{n}.$$

Regardless of λ_1, the variance tends to 0, as t increases. Ultimately, all individual budgets match up, despite investment differences at the start of the arms race. Indeed, the initial $x_i(0)$ investment is irrelevant to the long-term dynamics of $x_i(t)$. What the initial investment influences is the pace at which the variation in military spending is undertaken, which assigns unique behaviours to each actor. Taking the rate of change of the intermediate term of $x_i(t)$,

$$\frac{d}{dt}\left(\left(x_i(0) - \frac{\sum_j^n x_j(0)}{n} \right) \cdot \mathbb{e}^{(-b-a)\cdot t} \right) = (\mu(0) - x_i(0)) \cdot (a+b) \cdot \mathbb{e}^{(-b-a)\cdot t}.$$

In the situation where the i-th actor has a defence expenditure higher than the average of the initial investments of all the actors, there is an instant decrease in military expenditures, due to the preponderance of the actor with greater investment over the general average. However, it is important to note that, even here, the initial individual investment is related to a systemic variable, which is the average of the investments of all the actors, at time $t = 0$.

We should inquire to which domain of constants solution 4.1 holds, that is, it is strictly greater than zero for all positive time considered. After algebraic manipulations, we find,

$$\mu(0)(1-\mathbb{e}^{-nt}) + x_i(0)\mathbb{e}^{-nt} > \frac{c}{-b+(n-1)a}(\mathbb{e}^{b-(n-1)at} - 1)$$

It is of the form $f(t) > g(t)$, with both functions continuous and differentiable on $t \in \mathcal{T}$. Under such properties, we may encounter maximum and minimum values, which provide the appropriate domain for non degenerated cases,

$$\max(x_i(0), \mu(0)) \geq \begin{cases} \max(0, -\frac{c}{-b+(n-1)a}), \lambda_1 > 0 \\ \max(0, sgn(c)\cdot\infty), \lambda_1 < 0 \end{cases}$$

Where sgn denotes the sign function. At first, there are values of the constants $(x_i(0), \mu(0), a, b, c, n)$ that do not hold for the above inequality, however, when we consider $n \to \infty$, it is clear that $\lambda_1 \to \infty > 0$ and $-\frac{c}{-b+(n-1)a} \to 0$, which means that the inequality is satisfied for any set of valid $x_i(0) > 0$.

One interesting case arises when we consider the situation when domestic restraints match foreign incentives. However, we are unable to simply equate both terms, as it produces an undefined expression, since the denominator of some terms in equation 4.1 match 0. Still, treating temporarily constants as variables, we are able to inquire the limit as we approach this particular case, $-b+(n-1) \to a$. It is possible to find the explicit solution, given by,

$$\lim_{b\to(n-1)\cdot a} x_i(t) = \frac{\sum_i^n x_i(0)}{n} + c\cdot t + \left(x_i(0) - \frac{\sum_i^n x_i(0)}{n} \right) \cdot \mathbb{e}^{-a\cdot n\cdot t}. \tag{4.2}$$

As a limit situation, the solution when $\lambda_1 \to 0$ is not covered by the $\lambda_1 < 0$ or $\lambda_1 > 0$ cases, since the dominant term is ct. It is not surprising that, as the internal and external frequencies are the same, the exogenous factor explains the general trend of the model. For $c > 0$, spending tends to grow continuously, while for $c < 0$, investments decrease. For $c = 0$, $x_i(t)$ tends to the initial average of investments, the actors only balance the initial differences around this equilibrium point, which ends up being different from the original critical point in value, but not in behaviour. However, not all values of c are permitted. Following the same reasoning we applied on equation 4.1 on 4.2, we find that the condition on the constants is $\max(x_i(0), \mu(0)) \geq -ct$, which can only be achieved when $c \geq 0$, $\forall t \in \mathcal{T}$.

We should mention that the only stability condition for positive λ_1 is when the first term of $x_i(t)$ zeroes the dominant exponential, $\frac{\sum_j^n x_j(0)}{n} - \frac{c}{b-(n-1)\cdot a} = 0$. It is the condition in which the initial average of the system equals $\frac{c}{-b+(n-1)\cdot a}$, with just a redistribution of individual budgets towards the critical point, where it occurs the equality of budgets between the actors. However, it is important to highlight the difficulty in achieving stability in this case, as it is very unlikely that, in a continuous distribution of a system taken at random, the initial average will be equivalent to $-\frac{c}{b-(n-1)\cdot a}$, that is, it corresponds to a point, but, from a Lebesgue measure perspective on a probability space, individual points in a multidimensional Euclidean space have a measure of 0, therefore, the probability of randomly finding the initial system in such position is also 0.

For systemic inequality investigation, in the special case, the average is given by,

$$\lim_{b\to(n-1)\cdot a} \frac{\sum_i^n x_i(t)}{n} = \frac{\sum_j^n x_j(0)}{n} + c \cdot t$$

Nonetheless, $\sigma^2(t)$ still tends to 0, when $t \to \infty$.

5 General case: differentiated actors

We may tackle (3.4) not by searching exact solutions, but, rather, investigating ODE system's properties that are subject to analytical investigation. For sufficiently large shift $s^* \in \mathbb{R}_*^+$, $\mathbf{M}_{s^*} = \mathbf{M} + s^*\mathbf{I}$, $\mathbf{I}$ being the identity matrix of dimension n by n, $\mathbf{M}_s$ is as strictly positive matrix, that is, corresponding entries are strictly positive, therefore we may apply the following results on $\mathbf{M}_{s^*}$,

Theorem 5.1 (Perron–Frobenius theorem)**.** *"(a) If [matrix] A is positive, then [spectral radius] $\rho(A)$ is a simple eigenvalue, greater than the magnitude of any other eigenvalue. (b) If $A > 0$ is irreducible then $\rho(A)$ is a simple eigenvalue, any eigenvalue of A of the same modulus is also simple, A has a positive eigenvector x corresponding to $\rho(A)$, and any non-negative eigenvector of A is a multiple of x."*

Source: [4, p. 27]

Let r be the spectral radius of matrix A, s_i denote the sum of elements of the i-th row of A, $S = \max_i s_i$, and $s = \min_i s_i$, then,

Theorem 5.2. *Let $A \geq 0$ be irreducible. Let x be a positive eigenvector and let $\gamma = \max_{i,j}(x_i/x_j)$. Then,*

$$s \leq r \leq S$$

and

$$(S/s)^{1/2} \leq \gamma.$$

Source: [4, p. 37-38]

First, we need some additional notation. $b_i^* = a_{ii}$ and $c_i = b_i$. Moreover, we may organise entries $a_{i \neq j}$, such that $a_{i\neq j}^* \geq a_{i\neq k}^*, \forall j \geq k$. If $\lim_{n\to\infty} \lim_{j\to n^-} a_{i\neq j}^* \neq 0$, the impact of frequency of expenditures does not vanish as the number of actors increase, then $\lim_{n\to\infty} \lim_{j\to n^-} s_i = \lim_{n\to\infty} \lim_{j\to n^-} b_i^* + \sum_{j\neq i}^n a_{ij} + c_i - s^* = \infty > 0$, whenever s^* rate of change, as n increases, is inferior to that of the divergent series. However, there are limits to how large $|b_i|$ is, and by extension s^*, because b_i represents domestic disincentives proportional to already incurred military expenses. It is clear that $\mathbf{M}$ is irreducible, as it may not be written as a upper triangular matrix, since there are no 0 entries, therefore there is no permutation of rows and columns that produces upper triangular matrix and theorems (5.1) and (5.2) fully apply.

From theorem (5.2), considering $\lambda_{max} = r$,

$$\min\ b_i^* + \sum_{j\neq i}^n a_{ij} + c_i \leq \lambda_{max} - s^* \leq \max\ b_k^* + \sum_{l\neq k}^n a_{kl} + c_k$$

Since $\lambda_{max} \geq \min\ b_i^* + \sum_{j\neq i}^n a_{ij}^* + c_i - s^*$, we have,

$$\lambda_{max} \geq \lim_{n\to\infty} \min\ b_i^* + \sum_{j\neq i}^n a_{ij}^* + c_i - s^* = \infty > 0$$

With at least one positive eigenvalue, we know that the critical point is neither a sink nor a spiral sink, resulting in not assured stability, furthermore, theorem (5.1) guarantees the existence of eigenvector with strictly positive entries associated to λ_{max} that act as an attractor to points in $(\mathbb{R}_*^+)^n$. However, one should notice that $x_i'(0) = b_i^* x_i(0) + \sum_{j\neq i}^n a_{ij} x_j(0) + c_i$. Organising entries $a_{i\neq j} x_j(0)$, such that $a_{i\neq j}^* x_j^*(0) \geq a_{i\neq k}^* x_k^*(0), \forall j \geq k$, if $\lim_{n\to\infty} \lim_{j\to n^-} a_{i\neq j}^* x_j^*(0) \neq 0$, the initial impact of "speed" of budgets' variation does not vanish as the number of actors increase, then $\lim_{n\to\infty} x_i'(0) > 0$. Applying first order approximation for $\Delta \in \mathbb{R}^+$, $x_i(0+\Delta) \approx x_i(0) + \Delta(b_i^* x_i(0) + \sum_{j\neq i}^n a_{ij}^* x_j^*(0) + c_i)$, thus $\lim_{n\to\infty} x_i(\Delta) = \infty$. This approximation has as upper boundary the error $\frac{x_i''(\eta)\Delta^2}{2!}$, for $\eta \in [0, \Delta]$, that maximises the second order derivative [6, p. 350]. For successive approximations, $\lim_{t\to\infty} \lim_{n\to\infty} x_i(t) = \infty$. We may also see that, for each step, the military expenditure is strictly increasing, $x_i'(t) > 0$, therefore, for any initial set $x_i(0) > 0$, and constants that fit the non vanishing criteria, $x_i(t)$ is strictly positive and satisfies our requirement of no negative or zero defence investments.

The critical point $\mathbf{x}_c$ is given by Cramer's rule,

$$\mathbf{x}_c = \left(\frac{\begin{vmatrix} -c_1 & \dots & a_{1n} \\ \vdots & \ddots & \\ -c_n & & a_{nn} \end{vmatrix}}{\begin{vmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \\ a_{n1} & & a_{nn} \end{vmatrix}}, \dots, \frac{\begin{vmatrix} a_{11} & \dots & -c_1 \\ \vdots & \ddots & \\ a_{n1} & & -c_n \end{vmatrix}}{\begin{vmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \\ a_{n1} & & a_{nn} \end{vmatrix}} \right).$$

Applying Laplace's determinant expansion on the i-th element x_{ic} of the $\mathbf{x}_c$ vector, one finds,

$$\frac{\begin{vmatrix} a_{11} & \dots & -c_1 & & a_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{1n} & \dots & -c_n & \dots & a_{nn} \end{vmatrix}}{\begin{vmatrix} a_{11} & \dots & a_{1i} & & a_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{1n} & \dots & a_{ni} & \dots & a_{nn} \end{vmatrix}} = \frac{\sum_j^n -c_j(-1)^{i+j}det(M_{ij})}{\sum_j^n a_{ij}(-1)^{i+j}det(M_{ij})}. \tag{5.1}$$

However, we may interpret both dividend and divisor of (5.1) as scalar product between vectors, resulting in

$$\begin{aligned} \frac{\sum_j^n -c_j(-1)^{i+j}det(M_{ij})}{\sum_j^n a_{ij}(-1)^{i+j}det(M_{ij})} &= \frac{\mathbf{c} \cdot \mathbf{det}(\boldsymbol{\mathcal{M}}^*)}{\mathbf{a}_i \cdot \mathbf{det}(\boldsymbol{\mathcal{M}}^*)} \\ &= \frac{||\mathbf{c}|| ||\mathbf{det}(\boldsymbol{\mathcal{M}}^*)|| cos(\theta_{1i})}{||\mathbf{a}_i|| ||\mathbf{det}(\boldsymbol{\mathcal{M}}^*)|| cos(\theta_{2i})} \\ &= \frac{||\mathbf{c}|| cos(\theta_{1i})}{||\mathbf{a}_i|| cos(\theta_{2i})} \end{aligned}$$

Should $\lim_{n\to\infty} \frac{||\mathbf{c}||}{||\mathbf{a}_i||} = 0$ or $\cos(\theta_{1i}) = 0$, $\mathbf{x}_c = \mathbf{0}$, corresponding to the respective cases when the endogenous factors prevail over exogenous factors of the model and when there are no exogenous factors $\mathbf{c} = \mathbf{0}$, as, only then, the numerator of (5.1) is 0 for every entry of $\mathbf{x}_c$. Otherwise, there is no $\mathbf{x}_c \in (\mathbb{R}_*^+)^n$ with non-vanishing $\lim_{n\to\infty} \lim_{j\to n^-} a^*_{i\neq j} x^*_j \neq 0$, for $x_j \in \mathbb{R}_*^+$, that satisfies the linear equations system corresponding to (5.1).

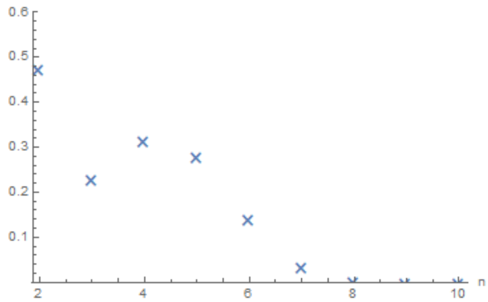

Figure 2: Frequency for the real part of all eigenvalues of $\boldsymbol{\mathcal{M}}$ being negative

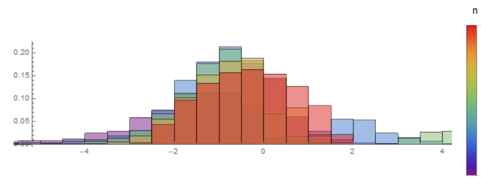

Figure 3: Histogram distribution for the real part of eigenvalues of $\frac{\boldsymbol{\mathcal{M}}}{\sqrt{n}}$

To help visualisation of the main results, we present the example of random generated $\boldsymbol{\mathcal{M}} = [(-1)^{\delta_{ij}} Lognormal(0,1)]$, with δ_{ij} being Kronecker's delta, in figures 2 and 3.

6 Final considerations

Richardson's model arises from deductive thinking, based on the reaction of how rational actors behave to face variations in military spending of the other components of the international system. While it has been relatively successful in matching reality with its application to the cases already mentioned in the early twentieth century and Cold War periods, the model has a number of limitations recognised in the literature. Criticising the ontological, methodological and epistemological foundations of the model exceeds our objectives; and are treated by [1], [5, p. 394-396] and [18, p. 246-247], so that we focus on the shortcomings that arise with its extension to the multivariable case.

The main limitation to be considered concerns the assumption of considering the constant matrix $\mathbf{A}$ and $\mathbf{M}$ for all actors in sections 3 and 4, fundamental for the analytical treatment of the deterministic system of ODEs. As it is a special case, it is not subject to analytical extension, as would be required in order to further the model's sophistication, introducing temporal variations through time dependent rates of budgetary expenditure, for example, seasonal trends that may affect actor's respective economies. Generalisation should be pursued in further works, despite there being no guarantees that analytical solutions will not face the same shortcomings we have found for the constant coefficients case, most notably, eigenvalues' determination for $n \geq 5$.

One of the limitations of the original Richardson model is that it presupposes that actors behave mechanically to changes in military stocks, that is, without *a priori* possibility of engaging in behaviour different from that described by system 3.1. Nikol'skii [19] seeks to avoid the situation by recourse to control theory, conceptualising expenditure as controllable by one of the actors in two different models, first, on the exogenous factor (linear model) and, second, rate of expenditure change in response to the opponent current level of military stock, offering general form and conditions of solution.

The paper [20] extends the previous work and makes explicit analytical solution for special case of constant coefficients. Both linear and bilinear, solution is either constant or step function, with only one discontinuity. Before commenting on this finding, we refer to [17] article, which may be thought as an extension of [19] linear controlled model, since it considers that not only one, but both actors have controls on the respective investments and engage in a differential game dynamic. Their approach differs from that of [19], because they consider, in addition to controls, a quadratic loss function to be minimised by the actors, but their conclusion partially converges with [20], as they find that the solution for the control is constant. Both [17] and [20] reach models that are nearly identical to the original Richardson work, but they provide important insight on arms race theorising, as "the coefficients of this set of equations are derived explicitly in terms of the parameters from each nation's optimization in the differential game" [17, p. 1139]. We are inclined to conjecture new hypothesis to be tested - does a similar conclusion happen in the bilinear case for both actors with controllable inputs in a differential game? Moreover, if analytical treatment exists, do the linear and bilinear controls behave similarly for the multivariate case?

Nevertheless, it is an interesting extension of the Richardson's model, as it highlights the effects of structural variables on the systemic behaviour. We should remember that both sides of the neo-neo debate place heavy emphasis on structure and draw extensively from rational choice theory, game theory and microeconomics. Despite such background, the IR field still lacks thorough mathematical theorising, being the Richardson's model one of the few exceptions. In this sense, we argue reasoning developed in the present paper is not altogether separated from the focal points of IR theory. According to [25], bipolar systems are the most stable, and our conclusion corroborate partially to this idea, as we have shown the destabilising effects caused by an increase in the number of actors, but other factors presented by Waltz are not captured by Richardson's model, such as perfectness and completion of information.

To summarise, the extension of the original case of the Richardson's model of arms race between two countries to multiple actors revealed new dynamics and explanatory possibilities, especially at the systemic level, of behaviour in international military spending. In this sense, it should be highlighted as the main contribution of the present paper the verification of the emergence of the issue of scale in the model, in which it was observed that the system cannot support an arbitrary number of actors, without losing stability.

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