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USING BERNOULLI NUMBERS TO GENERALIZE A LIMIT OF FINITE SUM ARISING FROM VOLUME COMPUTATIONS WITH THE SQUEEZE THEOREM

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Abstract

We developed in this work the computation of the volume of the sphere via the method of exhaustion by inscribed truncated right cones. We show that this computation can be used in calculus courses in several ways; mainly, to motivate and clarify the usage of the squeeze theorem in the computations of sum limits. As a result, we generalized a sum limit using Bernoulli numbers, producing a magnificent example of applied mathematics, and highlighting the importance of exploring when studying mathematics.
keywords: The method of exhaustion, sphere volume, combinatorics, calculus teaching, GeoGebra 3D.

Abstract

Neste trabalho, apresentamos o cálculo do volume da esfera através do método da exaustão inscrevendo troncos de cone retos. Mostramos que essa estratégia pode ser usada em cursos de cálculo de várias maneiras; principalmente, para motivar e esclarecer o uso do teorema do confronto no cálculo de limites de somas. Como resultado, generalizamos o limite de uma soma finita empregando os números de Bernoulli, produzindo um belo exemplo de matemática aplicada e evidenciando a importância de explorar quando se estuda matemática.
Palavras-chave: Método da Exaustão, volume da esfera, combinatória, ensino de cálculo, GeoGebra 3D.

1 Introduction

The teaching of calculus [4] and analysis become more significant when the professor employs representative problems to develop or illustrate ideas and concepts. Thus, to calculate limits of continuous functions, either by definition and properties [1, 22, 23] or by using the squeeze theorem [8], professors can investigate problems arising from physics, biology, and the various branches of engineering. Now to calculate limits of finite sums, geometry is a fertile field of problems to explore, such as the calculation of areas and volumes [18, 19] through the method of exhaustion.

Problems involving calculating areas and volumes permeate mathematics history [3]. The method of exhaustion [11, 20] is a technique invented by the classical Greeks to determine the area of a plane figure through the inscription and circumscription of a sequence of polygons whose sum of the areas is close to the area of the figure under study. Just like the area of a plane figure, we can also calculate the volume of a solid inscribing or circumscribing polyhedra or other solids, like cylinders for example, whose sum of volumes is close to the volume of the solid. However, the method of exhaustion has limitations in the calculation of areas and volumes [5].

Archimedes of Syracuse (287 BCE - 212 BCE) used the method of exhaustion to determine various mathematical results, such as the delimitation for the irrational constant π, the area of the parabolic segment, and the volume of the sphere [2, 3, 10]. As for the first result, Archimedes established $\frac{223}{71} < \pi < \frac{22}{7}$ by calculating the perimeter of regular polygons of up to 96 sides, respectively, inscribed and circumscribed to the circumference. For the second, he determined, inscribing and circumscribing rectangles, that area of the plane region bounded by the graph of the function $y = x^2$, the x axis, and the vertical lines $x = 0$ and $x = b$, is equal to $\frac{b^2}{3}$. As for the third, Archimedes established that the volume of a sphere is four times the volume of a cone with radius and height equal to the radius of the sphere.

The above-mentioned results "are widely recognized as ancient harbingers of the modern squeeze theorem" ([3], p. 56), presented in a version for sequences in Theorem 1, and whose demonstration can be found in Dunn [6] and Stewart [23].

Theorem 1.1 (squeeze theorem for sequences). *Let a_n, b_n and c_n numerical sequences such that $a_n \leq b_n \leq c_n$ for all $n \geq n_0$. If $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n$ exists and $\lim_{n\to\infty} b_n = L$.*

Following Archimedes' strategy, we can calculate the volume of the sphere using the method of exhaustion inscribing and circumscribing right cylinders, or inscribing truncated right cones. Using the latest strategy, we notice that this approach:

1. Gives us a very convenient context to understand the origins and intuitions of Theorem 1.1. In this concrete case, the presentation parallels the historical development in a technically simple framework.

2. Can be used as the starting point of exploration in several areas and viewpoints of mathematics, providing material and activities for the interested student to feel mathematics as an alive, dynamic subject.

In this way, this paper has two main characters: on the one hand, the computation of the volume of the sphere via the method of exhaustion by inscribed truncated right cones in a hemisphere; on the other hand, the use of the squeeze theorem to calculate sum limits arising from sphere volume computations. Concretely, we present:

- The standard method for inscribing/circumscribing cylinders to calculate the volume of the sphere, which exemplifies the historical Greek approach to these computations via the method of exhaustion, and produces the subject of computation of sums of squares and its rich collection of intuitions and methods.

- A more precise way of approximating the volume of the sphere by inscribing truncated right cones, that introduces the squeeze theorem in a rather natural way. In fact, in several ways, two of them, geometrical and another one algebraic, reflect the tension of geometric and algebraic approaches during the formalization of calculus ([3], chapter VII).

- The generalization of a limit resulting from the sphere volume computations, producing more sophisticated examples of the usage of the squeeze theorem and introducing the subject of sums of powers, Bernoulli numbers and its associated combinatorics.

2 Method

The motivating problem of this work was the use of the exhaustion method to determine the volume of the sphere - Theorem 2.1.

Theorem 2.1 (volume of the sphere)**.** *The volume $\mathcal{V}$ of the sphere ε with radius r is given by* $\mathcal{V}(\varepsilon) = \dfrac{4}{3}\pi r^3$.

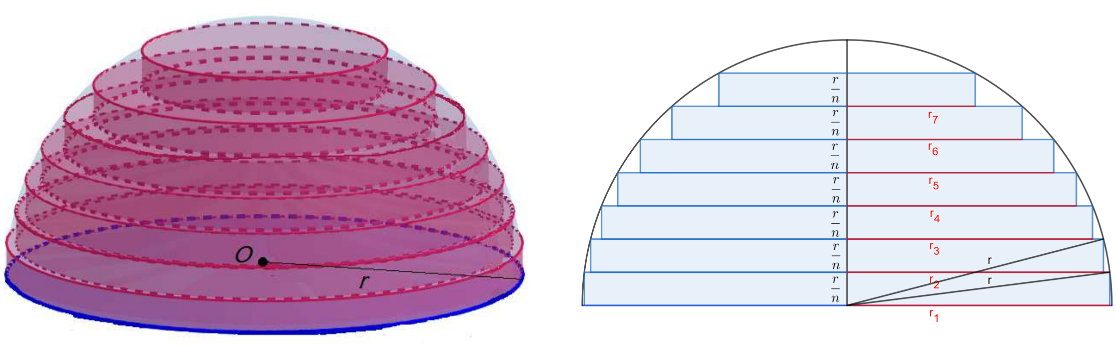
The standard approach in mathematical literature is to inscribe or circumscribe right cylinders in the sphere or in the hemisphere [11, 20, 24], which produces finite sum limits. These limits are calculated using the sum of the first k natural numbers.

In the investigative process, we decided to adopt a more precise strategy: inscribing truncated right cones in the hemisphere (the lateral surface of a truncated right cone fits better to the spherical surface than the lateral surface of a right cylinder). This approach generated a more complex limit, which we calculated using Theorem 1.1. The computational investigation of this limit (Appendix) helped us determine its generalization. In this sense, we work on one of the meanings of the generalization process in mathematics: to extend a mathematical concept or property [14]. We use during this process a dynamic geometry software, GeoGebra 3D [7] , to build a bidimensional and three-dimensional figures, which illustrate the old and new approaches.

2.1 The standard method: inscribing and circumscribing cylinders

Consider δ a hemisphere of radius r in which n right cylinders of height $\dfrac{r}{n}$ are inscribed, as shown in Figure 1(a).

Figure 1: Volume of the sphere by the method of exhaustion: (a) cylinders inscribed in the hemisphere; (b) radii of the cylinders inscribed in the hemisphere



(a) (b)

Source: The authors, with GeoGebra 3D.

In the inscription of n right cylinders, we should express the radius r_i, $i = 1, 2, \ldots, n$, of each cylinder as a function of the radius r of the hemisphere. For this purpose, it is sufficient to apply the Pythagorean theorem [13] to the rectangles defined in the meridian section of the hemisphere, as shown in Figure 1(b). In that way, we have

$$
\begin{aligned}
r_1^2 =& r^2 - \left(\frac{r}{n}\right)^2, \\
r_2^2 =& r^2 - \left(\frac{2r}{n}\right)^2, \\
r_3^2 =& r^2 - \left(\frac{3r}{n}\right)^2, \\
&\vdots \\
r_{n-1}^2 =& r^2 - \left[\frac{(n-1)r}{n}\right]^2, \\
r_n^2 =& r^2 - \left(\frac{nr}{n}\right)^2.
\end{aligned}
$$

Thus,

$$r_i^2 = r^2 - \left(\frac{ir}{n}\right)^2 = r^2\left(1 - \frac{i^2}{n^2}\right). \tag{2.1}$$

The volume of a right cylinder of radius R is $\pi R^2 h$ [11, 16], where h is the height of the cylinder. Thus, using (2.1) and $h = \frac{r}{n}$, the volume $\mathcal{V}_i$ of each cylinder inscribed in the hemisphere is equal to:

$$\mathcal{V}_i = \pi r^2\left(1 - \frac{i^2}{n^2}\right)\frac{r}{n} = \pi r^3\left(\frac{1}{n} - \frac{i^2}{n^3}\right). \tag{2.2}$$

In (2.2), $i = n$ represents a cylinder with $r_n = 0$ and volume $\mathcal{V}_n = 0$ (degenerate cylinder). The sum of the volumes of the n cylinders provides an approximation for the volume $\mathcal{V}(\delta)$ of the hemisphere. Thus, intuitively:

$$\mathcal{V}(\delta) \approx \sum_{i=1}^{n} \mathcal{V}_i. \tag{2.3}$$

Replacing (2.2) in approximation (2.3), and using the properties of a discrete sum, we obtain

$$\mathcal{V}(\delta) \approx \sum_{i=1}^{n} \pi r^3\left(\frac{1}{n} - \frac{i^2}{n^3}\right) = \pi r^3 \frac{1}{n^3}\sum_{i=1}^{n}\left(n^2 - i^2\right). \tag{2.4}$$

To improve the approximation (2.4), we can increase the value of n, i.e. we can inscribe a larger number of cylinders in the hemisphere. Thus, for $n \to \infty$, which implies $\frac{r}{n} \to 0$, we have

$$\mathcal{V}(\delta) = \lim_{n\to\infty} \sum_{i=1}^{n} \mathcal{V}_i = \lim_{n\to\infty} \pi r^3 \frac{1}{n^3} \sum_{i=1}^{n} \left(n^2 - i^2\right) = \pi r^3 \lim_{n\to\infty} \frac{1}{n^3} \sum_{i=1}^{n} \left(n^2 - i^2\right),$$
$$\mathcal{V}(\delta) = \pi r^3 \lim_{n\to\infty} \left(\frac{1}{n^3} \sum_{i=1}^{n} n^2 - \frac{1}{n^3} \sum_{i=1}^{n} i^2 \right) = \pi r^3 \lim_{n\to\infty} \left(1 - \frac{1}{n^3} \sum_{i=1}^{n} i^2 \right). \tag{2.5}$$

Here we have, naturally, the appearance of the sum of the squares of the first n natural numbers, whose closed formula [9, 12, 15] is given by

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}. \tag{2.6}$$

Then, using (2.6) in (2.5), we obtain

$$\lim_{n\to\infty} \frac{1}{n^3} \sum_{i=1}^{n} i^2 = \lim_{n\to\infty} \frac{1}{n^3} \left[\frac{n(n+1)(2n+1)}{6} \right],$$
$$\lim_{n\to\infty} \frac{1}{n^3} \sum_{i=1}^{n} i^2 = \lim_{n\to\infty} \frac{2n^3 + 3n^2 + n}{6n^3} = \lim_{n\to\infty} \left(\frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2} \right) = \frac{1}{3}. \tag{2.7}$$

Now, replacing (2.7) in (2.5), we conclude

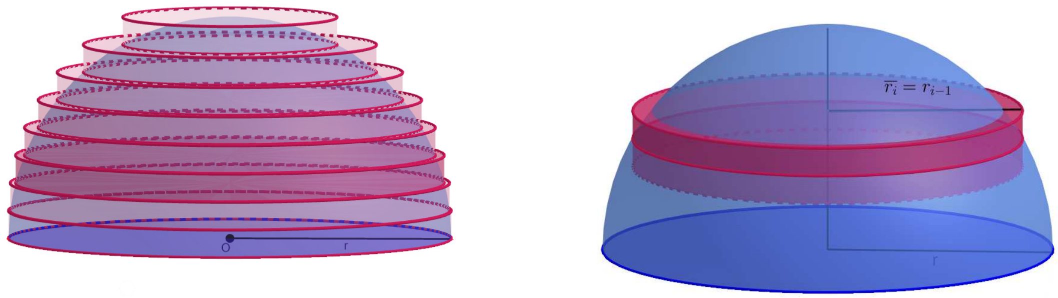
$$\mathcal{V}(\delta) = \frac{2}{3}\pi r^3. \tag{2.8}$$

In addition, multiplying (2.8) by two, we have the volume of the sphere given by Theorem 2.1.

To prove the volume of the sphere circumscribing right cylinders in a hemisphere, as illustrated in Figure 2(a), we repeat the beginning of the proof for inscribed cylinders. However, as can be seen in Figure 2(b), the radius of a given circumscribed cylinder is the same as the radius of the previous inscribed cylinder.

Then, denoting with bars the quantities related to the circumscribed cylinders, we

Figure 2: Volume of the sphere by the method of exhaustion: (a) cylinders circumscribed in the hemisphere; (b) comparison between the radii of the circumscribed and inscribed cylinder in the hemisphere



(a) (b)

Source: The authors, with GeoGebra 3D.

get

$$\begin{aligned}
\bar{r}_1^2 =& r^2, \\
\bar{r}_i^2 =& r^2 - \left[\frac{(i-1)r}{n}\right]^2 = r^2\left[1 - \frac{(i-1)^2}{n^2}\right],\ i = 2, \dots, n, \\
\bar{\mathcal{V}}_i =& \pi r^2 \left[1 - \frac{(i-1)^2}{n^2}\right]\frac{r}{n} = \pi r^3\left[\frac{1}{n} - \frac{(i-1)^2}{n^3}\right], \\
\mathcal{V}(\delta) \approx \sum_{i=1}^{n} \bar{\mathcal{V}}_i =& \sum_{i=1}^{n} \pi r^3 \left[\frac{1}{n} - \frac{(i-1)^2}{n^3}\right] = \pi r^3 \frac{1}{n^3}\sum_{i=1}^{n}\left[n^2 - (i-1)^2\right], \\
\mathcal{V}(\delta) = \lim_{n\to\infty}\sum_{i=1}^{n}\bar{\mathcal{V}}_i =& \lim_{n\to\infty}\pi r^3 \frac{1}{n^3}\sum_{i=1}^{n}\left[n^2 - (i-1)^2\right],
\end{aligned}$$

$$\mathcal{V}(\delta) = \pi r^3 \lim_{n\to\infty}\left[1 - \frac{1}{n^3}\sum_{i=1}^{n}(i-1)^2\right]. \tag{2.9}$$

Now we can change variables in the sum in (2.9). Setting $j = i - 1$, we have

$$\sum_{i=1}^{n}(i-1)^2 = \left(\sum_{j=1}^{n} j^2\right) - n^2. \tag{2.10}$$

Thence, using (2.7) and (2.10) in (2.9)

$$\mathcal{V}(\delta) = \pi r^3 \lim_{n\to\infty} \left[1 - \frac{1}{n^3}\left(\sum_{j=1}^{n} j^2\right) + \frac{1}{n}\right] = \frac{2}{3}\pi r^3.$$

In addition, we have the same limit $\mathcal{V}(\varepsilon) = \frac{4}{3}\pi r^3$ for the volume of the sphere, and due to these computations, we get an estimate of the cumulative difference in the volumes of the inscribed and circumscribed cylinders:

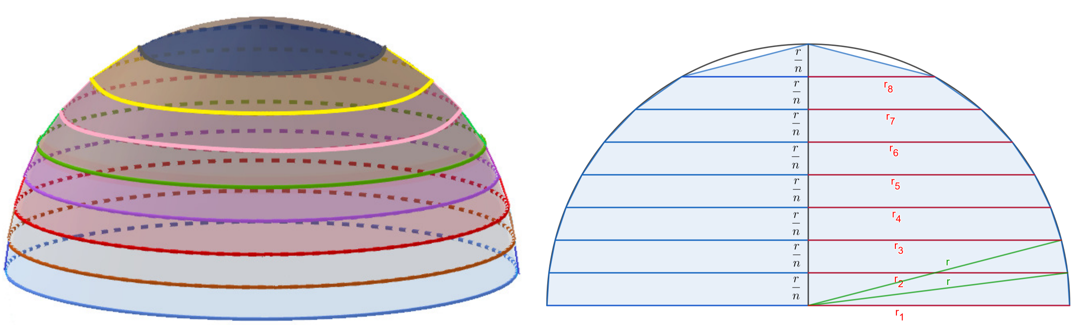
$$\sum_{i=1}^{n} \left(\bar{\mathcal{V}}_i - \mathcal{V}_i\right) = \bar{\mathcal{V}}_1 - \mathcal{V}_n = \pi r^3 \frac{1}{n}. \tag{2.11}$$

The two sums appearing in (2.11) are actually the same sum with different summation limits.

2.2 Inscribing truncated right cones

Let us consider δ a hemisphere of radius r in which n truncated right cones of height $\frac{r}{n}$ are inscribed, as shown in Figure 3(a).

Figure 3: Volume of the sphere by the method of exhaustion: (a) truncated right cones inscribed in the hemisphere; (b) radii of the truncated right cones inscribed in the hemisphere



(a) (b)

Source: The authors, with GeoGebra 3D.

In the inscription of n truncated right cones, we should express the radius $\hat{r}_i$, $i = 1, 2, \ldots, n, n+1$, for each bases of the truncated cone as a function of the radius r of the hemisphere. For this goal, it is sufficient to apply the Pythagorean theorem to the trapezoids defined in the meridian section of the hemisphere, as shown in Figure 3(b). In that way, we have

$$\begin{aligned}
\hat{r}_1^2 =& r^2, \\
\hat{r}_2^2 =& r^2 - \left(\frac{r}{n}\right)^2, \\
\hat{r}_3^2 =& r^2 - \left(\frac{2r}{n}\right)^2, \\
\hat{r}_4^2 =& r^2 - \left(\frac{3r}{n}\right)^2, \\
&\vdots \\
\hat{r}_{n-1}^2 =& r^2 - \left[\frac{(n-2)r}{n}\right]^2, \\
\hat{r}_n^2 =& r^2 - \left[\frac{(n-1)r}{n}\right]^2, \\
\hat{r}_{n+1}^2 =& r^2 - \left(\frac{nr}{n}\right)^2.
\end{aligned}$$

Thus,

$$\hat{r}_i^2 = r^2 - \left[\frac{(i-1)r}{n}\right]^2 = r^2\left[1 - \frac{(i-1)^2}{n^2}\right], \tag{2.12}$$

$$\hat{r}_i = \frac{r}{n}\sqrt{n^2 - (i-1)^2}, \text{ with } i = 1, 2, \ldots, n+1. \tag{2.13}$$

The volume of a truncated right cone is $\frac{\pi h}{3}\left(\tilde{R}^2 + \tilde{R}\tilde{r} + \tilde{r}^2\right)$ [16], where h is the height, and $\tilde{R}$ and $\tilde{r}$ are the radius of the basis of the truncated right cone. Thus, the volume $\hat{\mathcal{V}}_i$ of each truncated right cone inscribed in the hemisphere is equal to

$$\hat{\mathcal{V}}_i = \frac{\pi}{3}\frac{r}{n}\left(\hat{r}_i^2 + \hat{r}_i\hat{r}_{i+1} + \hat{r}_{i+1}^2\right),$$

which after using (2.12), and (2.13), and $h = \frac{r}{n}$, and some algebraic manipulations transforms to

$$\hat{\mathcal{V}}_i = \frac{\pi}{3}r^3\left\{\frac{1}{n^3}\left[n^2 - (i-1)^2\right] + \frac{1}{n^3}\sqrt{(n^2 - i^2)\left[n^2 - (i-1)^2\right]} + \frac{1}{n^3}\left(n^2 - i^2\right)\right\}. \tag{2.14}$$

In (2.14), $i = n$ represents a right cone with $\hat{r}_{n+1} = 0$ and volume $\hat{\mathcal{V}}_n = \frac{1}{3}\pi\hat{r}_n^2\frac{r}{n}$ (degenerate truncated right cone).

The sum of the volumes of the n truncated right cones provides an approximation for the volume $\mathcal{V}(\delta)$ of the hemisphere. Thus, intuitively,

$$\begin{aligned}\mathcal{V}(\delta) &\approx \sum_{i=1}^{n}\hat{\mathcal{V}}_i,\\ \mathcal{V}(\delta) &\approx \frac{\pi}{3}r^3\frac{1}{n^3}\sum_{i=1}^{n}\left\{\left[n^2-(i-1)^2\right]+\sqrt{(n^2-i^2)\left[n^2-(i-1)^2\right]}+\left(n^2-i^2\right)\right\}. \end{aligned}\tag{2.15}$$

Therefore, the limit (if it exists)

$$\begin{aligned}\mathcal{V}(\delta) &= \lim_{n\to\infty}\sum_{i=1}^{n}\hat{\mathcal{V}}_i,\\ \mathcal{V}(\delta) &= \frac{\pi}{3}r^3\lim_{n\to\infty}\frac{1}{n^3}\sum_{i=1}^{n}\left\{\left[n^2-(i-1)^2\right]+\sqrt{(n^2-i^2)\left[n^2-(i-1)^2\right]}+\left(n^2-i^2\right)\right\}, \end{aligned}\tag{2.16}$$

represents the volume of the hemisphere δ.

We now proceed to study the limit in (2.16) from progressively finer viewpoints.

3 Results

Notice that the expression (2.16) has three summands, one of them containing the square root, and the other two having been computed in Section 2. Thus if one can compute

$$\lim_{n\to\infty}\frac{1}{n^3}\sum_{i=1}^{n}\sqrt{(n^2-i^2)\left[n^2-(i-1)^2\right]},\tag{3.1}$$

then it is done. We shall see that this square root term increases several interesting developments.

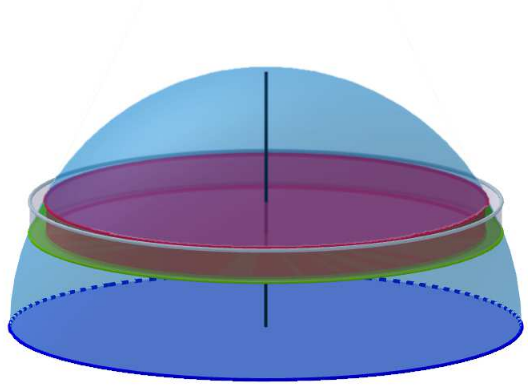
The key observation to be made here is that each summand in (2.15) represents the volume of an inscribed truncated right cone. As such, inspecting Figure 4 shows that

$$\mathcal{V}_i \leq \hat{\mathcal{V}}_i \leq \bar{\mathcal{V}}_i.\tag{3.2}$$

Then

$$\sum_{i=1}^{n}\mathcal{V}_i \leq \sum_{i=1}^{n}\hat{\mathcal{V}}_i \leq \sum_{i=1}^{n}\bar{\mathcal{V}}_i.$$

Figure 4: Volume of the sphere by the method of exhaustion: comparison between inscribed cylinder, inscribed right cone, and circumscribed cylinder



Source: The authors, with GeoGebra 3D.

Now, we can apply the squeeze theorem (Theorem 1.1). Since we have computed in Section 2 the external limits

$$\lim_{n\to\infty}\sum_{i=1}^{n}\mathcal{V}_i,\ \ \lim_{n\to\infty}\sum_{i=1}^{n}\bar{\mathcal{V}}_i,$$

and they have the same value, it follows that

$$\mathcal{V}(\delta)=\lim_{n\to\infty}\sum_{i=1}^{n}\hat{\mathcal{V}}_i=\frac{2}{3}\pi r^3.$$

Up to here, we have given a natural geometrical illustration of how the squeeze theorem works, thus providing a way of overcoming the pedagogical problem of "just another technique to memorize". Indeed, we have shown that it is not necessary to compute the limit (3.1). However, both this limit and the complete expression of the volume of each truncated right cone deserve further scrutiny. To begin with, note that the volume $\hat{\mathcal{V}}_i$ in (2.14) is a sum of three terms

$$\hat{\mathcal{V}}_i=\frac{1}{3}\left(\bar{\mathcal{V}}_i+\mathcal{B}_i+\mathcal{V}_i\right), \tag{3.3}$$

where

$$\mathcal{B}_i=\pi r^3\frac{1}{n^3}\sqrt{\left(n^2-i^2\right)\left[n^2-(i-1)^2\right]},$$

and

$$\lim_{n\to\infty} \frac{1}{n^3} \sum_{i=1}^{n} \sqrt{(n^2 - i^2)\left[n^2 - (i-1)^2\right]}$$

is the general term limit we are striving to compute. Note that the factor $\frac{1}{3}$ in front of the right side in (3.3) suggests that we are dealing with a mean. So, the inequalities in (3.2) indicate (but do not prove!) $\mathcal{V}_i \leq \mathcal{B}_i \leq \bar{\mathcal{V}}_i$, providing an avenue to apply the squeeze theorem directly to the square room term. We develop this strategy in two steps: geometric and algebraic.

3.1 The limit (3.1) by geometric bounds

A simple geometric/inequality argument shows that $\mathcal{V}_i \leq \mathcal{B}_i \leq \bar{\mathcal{V}}_i$ is true enough to apply the squeeze theorem. Indeed, from the inclusion of the inscribed right cylinder in the inscribed truncated right cone in the circumscribed right cylinder, we get as in (3.2)

$$\mathcal{V}_i \leq \frac{1}{3}\left(\bar{\mathcal{V}}_i + \mathcal{B}_i + \mathcal{V}_i\right) \leq \bar{\mathcal{V}}_i,$$

which we manipulate as follows:

$$\begin{aligned} 3\mathcal{V}_i \leq& \bar{\mathcal{V}}_i + \mathcal{B}_i + \mathcal{V}_i \leq 3\bar{\mathcal{V}}_i; \\ 2\mathcal{V}_i - \bar{\mathcal{V}}_i \leq& \mathcal{B}_i \leq 2\bar{\mathcal{V}}_i - \mathcal{V}_i; \\ \mathcal{V}_i + \left(\mathcal{V}_i - \bar{\mathcal{V}}_i\right) \leq& \mathcal{B}_i \leq \bar{\mathcal{V}}_i + \left(\bar{\mathcal{V}}_i - \mathcal{V}_i\right). \end{aligned} \tag{3.4}$$

Therefore, we did not prove that $\mathcal{V}_i \leq \mathcal{B}_i \leq \bar{\mathcal{V}}_i$, but the previous argument shows that this inequality perturbed by the difference term $\mathcal{V}_i - \bar{\mathcal{V}}_i$ holds. Summing (3.4) in i, we have

$$\sum_{i=1}^{n} \mathcal{V}_i + \sum_{i=1}^{n} \left(\mathcal{V}_i - \bar{\mathcal{V}}_i\right) \leq \sum_{i=1}^{n} \mathcal{B}_i \leq \sum_{i=1}^{n} \bar{\mathcal{V}}_i + \sum_{i=1}^{n} \left(\bar{\mathcal{V}}_i - \mathcal{V}_i\right). \tag{3.5}$$

At this point, applying the difference formula (2.11) in (3.5), we obtain

$$\sum_{i=1}^{n} \mathcal{V}_i - \pi r^3 \frac{1}{n} \leq \sum_{i=1}^{n} \mathcal{B}_i \leq \sum_{i=1}^{n} \bar{\mathcal{V}}_i + \pi r^3 \frac{1}{n}. \tag{3.6}$$

Now we can apply the squeeze theorem to the limit (3.1) since $\lim\limits_{n\to\infty} \frac{1}{n} = 0$. Indeed,

$$\lim_{n\to\infty}\left(\sum_{i=1}^{n} \mathcal{V}_i - \pi r^3 \frac{1}{n}\right) = \frac{2}{3}\pi r^3,$$
$$\lim_{n\to\infty}\left(\sum_{i=1}^{n} \bar{\mathcal{V}}_i + \pi r^3 \frac{1}{n}\right) = \frac{2}{3}\pi r^3,$$

and inequality (3.6) allows us to apply the squeeze theorem to get

$$\lim_{n\to\infty} \sum_{i=1}^{n} \mathcal{B}_i = \frac{2}{3}\pi r^3.$$

Hence,

$$\lim_{n\to\infty} \frac{1}{n^3} \sum_{i=1}^{n} \sqrt{(n^2 - i^2)\left[n^2 - (i-1)^2\right]} = \frac{2}{3}.$$

3.2 The limit (3.1) by algebraic bounds

The previous geometric argument shows the importance of the "slack" given by terms with limit zero in the squeeze theorem. However, here we present that a more precise algebraic study of the inner limit shows that the exact inequality $\mathcal{V}_i \leq \mathcal{B}_i \leq \bar{\mathcal{V}}_i$ does hold. We have

$$-(i-1)^2 > -i^2, \tag{3.7}$$

for $1 \leq i \leq n$.

Adding n^2 on both sides of the inequality (3.7), we get

$$n^2 - (i-1)^2 \geq n^2 - i^2. \tag{3.8}$$

Multiplying both sides of the inequality (3.8) by $(n^2 - i^2)$ (note that $n^2 - i^2 > 0$), we obtain

$$\left(n^2 - i^2\right)\left[n^2 - (i-1)^2\right] \;\geq\; \left(n^2 - i^2\right)^2,$$
$$\sqrt{(n^2 - i^2)\left[n^2 - (i-1)^2\right]} \;\geq\; n^2 - i^2. \tag{3.9}$$

The other bound calculation is similar: multiplying both sides of the inequality (3.8) by the positive value $[n^2 - (i-1)^2]$, we get

$$\left[n^2 - (i-1)^2\right]^2 > \left(n^2 - i^2\right)\left[n^2 - (i-1)^2\right)],$$
$$\sqrt{(n^2 - i^2)\left[n^2 - (i-1)^2\right]} < n^2 - (i-1)^2. \tag{3.10}$$

Note that since $i \leq n$, the inequality (3.10) is easily shown to be strict.

From the inequalities (3.9) and (3.10), we have

$$\frac{1}{n^3}\sum_{i=1}^{n}\left(n^2-i^2\right) \leq \frac{1}{n^3}\sum_{i=1}^{n}\sqrt{(n^2-i^2)\left[n^2-(i-1)^2\right]} < \frac{1}{n^3}\sum_{i=1}^{n}\left[n^2-(i-1)^2\right]. \quad (3.11)$$

Using

$$\lim_{n\to\infty}\frac{1}{n^3}\sum_{i=1}^{n}\left(n^2-i^2\right) = \lim_{n\to\infty}\frac{1}{n^3}\sum_{i=1}^{n}\left[n^2-(i-1)^2\right] = \frac{2}{3}, \quad (3.12)$$

shown in Section 2, the inequality (3.11) allows the usage of the squeeze theorem, and we conclude

$$\lim_{n\to\infty}\frac{1}{n^3}\sum_{i=1}^{n}\sqrt{(n^2-i^2)\left[n^2-(i-1)^2\right]} = \frac{2}{3}.$$

3.3 Generalizing the limit (3.1)

The path we have followed allows us to apply the squeeze theorem for the limit

$$\lim_{n\to\infty}\frac{1}{n^{k+1}}\sum_{i=1}^{n}\sqrt{(n^k-i^k)\left[n^k-(i-1)^k\right]} = \frac{k}{k+1},$$

which generalizes the limit (3.1), but in this case we do not have any simple geometric interpretation that facilitates the computation of the limit. The key tools used to apply the squeeze theorem to compute the limit (3.1) were:

- The inequalities (3.11).
- The equality of the limits in (3.12), whose computation was made possible by the closed formula for the sum of the squares of the first n natural numbers.

The generalization of (3.11) is a direct adaptation of the proof using algebraic bounds. What we want to emphasize is that to compute the limits that generalize (3.12), we are introduced to the rich world of closed formulas for the sum powers [25] and Bernoulli numbers [21]. We describe the bare minimum needed for conducting the computations, but there is much more to be explored in this area.

Following Scharlau and Opolka [21], we define the Bernoulli numbers by the recurrence $B_0 = 1$ and

$$B_i = \sum_{s=0}^{i-1} B_s \frac{1}{s!}\frac{i!}{(i-s+1)!}.$$

The property of the Bernoulli numbers that we use is that the sum of k-th powers of the first $n-1$ integers,

$$S_k(n) = 1^k + 2^k + \cdots + (n-1)^k = \sum_{i=1}^{n-1} i^k,$$

can be expressed as a polynomial in n of degree $k+1$, whose coefficients can be expressed in terms of Bernoulli numbers by

$$S_k(n) = \frac{1}{k+1} \sum_{j=0}^{k} \binom{k+1}{j+1} B_{k-j} n^{j+1}.$$

Proposition 3.1. *Let i, n and k be positive integers, with $k \geq 2$. Then*

$$\lim_{n\to\infty} \frac{1}{n^{k+1}} \sum_{i=1}^{n} \sqrt{\left(n^k - i^k\right)\left[n^k - (i-1)^k\right]} = \frac{k}{k+1}.$$

Proof. The first step of the proof are the inequalities

$$\frac{1}{n^{k+1}} \sum_{i=1}^{n} \left(n^k - i^k\right) < \frac{1}{n^{k+1}} \sum_{i=1}^{n} \sqrt{\left(n^k - i^k\right)\left[n^k - (i-1)^k\right]},$$

$$\frac{1}{n^{k+1}} \sum_{i=1}^{n} \sqrt{\left(n^k - i^k\right)\left[n^k - (i-1)^k\right]} < \frac{1}{n^{k+1}} \sum_{i=1}^{n} \left[n^k - (i-1)^k\right].$$

To prove these inequalities, we proceed exactly as in the case of algebraic bounds, simply by replacing 2 with k everywhere. The nice part is then the outer limits, i.e. we want to show that

$$\lim_{n\to\infty} \frac{1}{n^{k+1}} \sum_{i=1}^{n} \left(n^k - i^k\right) = \lim_{n\to\infty} \frac{1}{n^{k+1}} \sum_{i=1}^{n} \left[n^k - (i-1)^k\right] = \frac{k}{k+1}.$$

In addition, we want to apply the squeeze theorem. For that, we know

$$\frac{1}{n^{k+1}} \sum_{i=1}^{n} \left(n^k - i^k\right) = \frac{1}{n^{k+1}} \sum_{i=1}^{n} n^k - \frac{1}{n^{k+1}} \sum_{i=1}^{n} i^k,$$

$$\frac{1}{n^{k+1}} \sum_{i=1}^{n} \left(n^k - i^k\right) = \frac{1}{n^{k+1}} n n^k - \frac{1}{n^{k+1}} \sum_{i=1}^{n} i^k = 1 - \frac{1}{n^{k+1}} \sum_{i=1}^{n} i^k.$$

A way to show that

$$\frac{1}{n^{k+1}}\sum_{i=1}^{n} i^k = \frac{1}{k+1}$$

is to use the expression of the sum $\sum_{i=1}^{n} i^k$ in terms of the Bernoulli numbers, i.e.

$$\sum_{i=1}^{n} i^k = \frac{1}{k+1}\sum_{j=0}^{k}\binom{k+1}{j+1} B_{k-j}(n+1)^{j+1}.$$

Therefore,

$$\begin{aligned}
\lim_{n\to\infty}\frac{1}{n^{k+1}}\sum_{i=1}^{n} i^k &= \lim_{n\to\infty}\frac{1}{n^{k+1}}\frac{1}{k+1}\sum_{j=0}^{k}\binom{k+1}{j+1} B_{k-j}(n+1)^{j+1},\\
\lim_{n\to\infty}\frac{1}{n^{k+1}}\sum_{i=1}^{n} i^k &= \frac{1}{k+1}\lim_{n\to\infty}\sum_{j=0}^{k}\binom{k+1}{j+1} B_{k-j}\frac{(n+1)^{j+1}}{n^{k+1}} = \frac{1}{k+1}, \qquad (3.13)\\
\lim_{n\to\infty}\frac{1}{n^{k+1}}\sum_{i=1}^{n}\left(n^k - i^k\right) &= \lim_{n\to\infty}\left(1 - \frac{1}{n^{k+1}}\sum_{i=1}^{n} i^k\right) = 1 - \frac{1}{k+1} = \frac{k}{k+1}.
\end{aligned}$$

The only term that survives the limit operation in (3.13) is the last. Therefore, we should know only the first Bernoulli number, whose combinatorics are much easier to grasp.

To prove

$$\lim_{n\to\infty}\frac{1}{n^{k+1}}\sum_{i=1}^{n}\left[n^k - (i-1)^k\right] = \frac{k}{k+1},$$

we first proceed in the following way:

$$\begin{aligned}
\frac{1}{n^{k+1}}\sum_{i=1}^{n}\left[n^k - (i-1)^k\right] &= \frac{1}{n^{k+1}}\sum_{i=1}^{n} n^k - \frac{1}{n^{k+1}}\sum_{i=1}^{n}(i-1)^k;\\
\frac{1}{n^{k+1}}\sum_{i=1}^{n}\left[n^k - (i-1)^k\right] &= \frac{1}{n^{k+1}} n n^k - \frac{1}{n^{k+1}}\sum_{i=1}^{n}(i-1)^k = 1 - \frac{1}{n^{k+1}}\sum_{i=1}^{n}(i-1)^k.
\end{aligned}$$

Again this computation benefit from the change of variable $j = i - 1$ and we get

$$\sum_{i=1}^{n}(i-1)^k = \left(\sum_{j=1}^{n} j^k\right) - n^k.$$

Finally, we conclude

$$\lim_{n\to\infty} \frac{1}{n^{k+1}} \sum_{i=1}^{n} \left[n^k - (i-1)^k\right] = \lim_{n\to\infty} \left[1 - \frac{1}{n^{k+1}} \left(\sum_{j=1}^{n} j^k\right) - \frac{1}{n}\right] = \frac{k}{k+1}.$$

□

4 Discussion

Our study showed that, to prove the volume of the sphere by the exhaustion method inscribing truncated right cones, we remain dependent on the limits established by the inscription and circumscription of right cylinders. In this sense, our methodology is not innovative. However, by computing the limit arising from the inscription of truncated right cones, we succeed generalized a more complex limit, thus producing a wonderful example of applied mathematics and evidencing that the calculation of volumes can be used to contextualize the determination of finite sum limits.

5 Concluding remarks

We expect that this study shows how considering a standard example (the volume of the sphere via the method of exhaustion by right cylinders) in a way that goes beyond the treatment usually found in a standard textbook, one can accomplish several valuable objectives: clarify the conceptual, technical and historical context of mathematical tools, in this case, the squeeze theorem; intertwine several areas of mathematics around the example given, in this case, geometry, analysis and combinatorics; motivate the importance of exploring when studying mathematics.

6 Recommendations

Our work motivates the study of the wonderful world of inequalities, which starting place can be the book of Niven [17], and the Bernoulli numbers and their associated combinatorics.

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Appendix

We used the C language code described below to investigate the sum limit

$$\lim_{n\to\infty} \frac{1}{n^{k+1}} \sum_{i=1}^{n} \sqrt{(n^k - i^k)\left[n^k - (i-1)^k\right]}. \tag{6.1}$$

```
#include <windows.h>
#include <stdio.h>
#include <stdlib.h>
#include <math.h>
int main()
{
  FILE*f=fopen("serie.txt","w");
  int i,k=2,n;
  double sum=0., sumaux=1.0;
  for(n=2;sumaux!=sum;n++){
    sumaux=sum;
    for(i=1;i<=n;i++){
      sum+=sqrt(((pow(n,k)-pow(i-1,k))*(pow(n,k)-pow(i,k))));
    }
    sum=sum/pow(n,k+1);
    fprintf(f,"n=%10d sum=%30.18g\n",n,sum);
  }
  fclose(f);
  return 0;
}
```

The Table 1 shows the numerical results for various values of k.

Table 1: Numerical results for the limit (6.1)

k	n	sum
2	150163	$0.66666666649773287 \approx \frac{2}{3}$
3	75154	$0.74999999906868420 \approx \frac{3}{4}$
4	80684	$0.79999999894431617 \approx \frac{4}{5}$
5	87662	$0.83333333222991424 \approx \frac{5}{6}$
9	97469	$0.89999999845799994 \approx \frac{9}{10}$
1000		Underflow error

Source: The authors.