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A DYNAMIC VIEW OF SOME GEOMETRIC LOCI VIA GEOGEBRA

Rudimar Nós
Universidade Tecnológica Federal do Paraná
rudimarnos@gmail.com

Mari Sano
Universidade Tecnológica Federal do Paraná
marisano@utfpr.edu.br

Victoria Mazotti Rodrigues da Silva
Universidade Tecnológica Federal do Paraná
victoriamazotti@gmail.com

Abstract

We present in this work dynamic experiences that explore interesting geometric loci, such as Miquel points, Fermat points, Nagel points, and Gergonne points. In these dynamic approaches, accessible through external links to pages on the GeoGebra platform, readers can change parameters through sliders and observe, as the parameters vary, whether the introduced changes satisfy the hypotheses that define the geometric loci. Additionally, we utilize GeoGebra to create two-dimensional geometric figures. In addition to providing definitions, we offer proofs of the theorems that establish the uniqueness of the covered geometric loci. Some of the proofs introduced in this work involve the concept of isotomic cevians, which needs more coverage in the existing literature. In summary, GeoGebra is an invaluable tool for constructing dynamic approaches to explore geometric loci. It empowers students to test hypotheses both before and after formal demonstrations. One concludes that GeoGebra is a versatile software that can be effectively integrated into geometry education at all levels.

Keywords: Isotomic cevians, Miquel's point, Fermat's points, Gergonne's point, Nagel's point.

Resumo

Apresentamos neste trabalho experiências dinâmicas que exploram lugares geométricos interessantes, como os pontos de Miquel, Fermat, Nagel e Gergonne. Nas abordagens dinâmicas, construídas em páginas da plataforma GeoGebra e acessadas por links externos, o leitor pode alterar parâmetros por meio de controles deslizantes e observar, com o movimento, se as alterações introduzidas atendem às hipóteses que definem os lugares geométricos. Também usamos o GeoGebra para construir figuras bidimensionais. Além das definições, apresentamos as demonstrações dos teoremas que garantem a unicidade do lugar geométrico abordado. Algumas das provas fornecidas usam noções de cevianas isotômicas, sendo que a literatura relacionada a esses conceitos é escassa. Concluímos que o GeoGebra é uma excelente ferramenta para a construção de abordagens dinâmicas de lugares geométricos, pois permite que os estudantes testem hipóteses antes ou depois de demonstrações formais. Dessa forma, o GeoGebra é um software que pode ser explorado pelo professor de geometria em todos os níveis de ensino.

Palavras-chave: Cevianas isotômicas, ponto de Miquel, ponto de Fermat, ponto de Gergonne, ponto de Nagel.

1 Introduction

In geometry, a geometric locus is a set of points that satisfy a specific property. Usually, a geometric shape or figure can be defined as a geometric locus.

Definition 1.1. Given a property P relative to a set of points in $\mathbb{R}^2$ or $\mathbb{R}^3$, the geometric locus of the points that possess property P is the subset L that satisfies the conditions:

1. every point in L possesses property P;

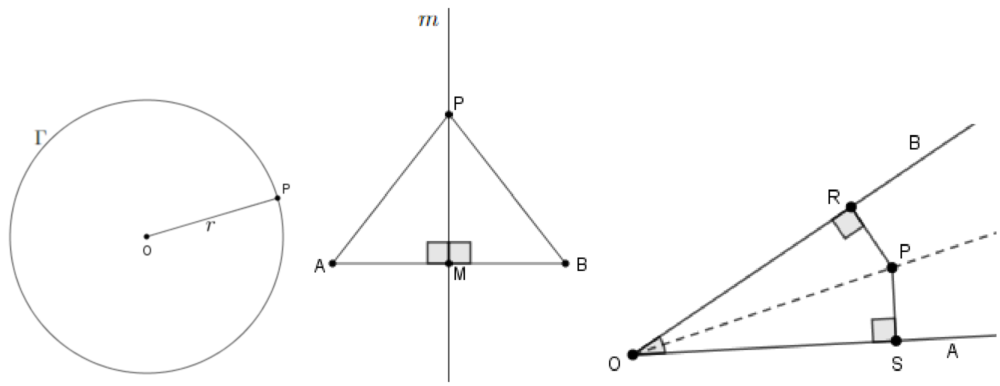
2. every point that possesses property P belongs to L.

Some elementary geometric loci, such as the circumference, the perpendicular bisector of a line segment, and the angle bisector, illustrated in Figure 1 [1], are easily grasped and visualized.

For more complex geometric loci, it can be a non-trivial task to visualize and characterize the set of points that satisfy the property. In any case, one can use digital technologies to visualize and explore the given geometrical situation. As articulated by Arcavi [1, p. 216]:

> Such "technologies" might develop visual means to better "see" mathematical concepts and ideas. Mathematics, as a human and cultural creation dealing with objects and entities quite different from physical phenomena (like planets or blood cells), relies heavily (possibly much more than mathematicians would be willing to admit) on visualization in its different forms and at different levels, far beyond the obviously visual field of geometry, and spatial visualization.

[1] All figures were produced by the authors with GeoGebra.



(a) (b) (c)

Figure 1: Geometric loci: (a) circumference Γ; (b) bisector m of the line segment AB; (c) bisector OP of the angle $\angle AOB$.

In the present work, we carry out the program of leveraging digital tools for the exploration of selected interesting geometric loci:

- The geometric constructions explored are the Miquel, Fermat, Gergonne, and Nagel points. The common theme of these constructions is that they are given by the unique solution of a specific concurrency situation; the Fermat point is also a solution to a significant minimization problem (see Section 3); in the proof of the theorem regarding the Nagel point, we use concepts related to isotomic cevians [20].

- The tool of choice for the present paper is GeoGebra software [8]. This tool allows for geometric constructions to be dynamically created and modified interactively. It enables merging elements such as figures, text, and equations, which are modified whenever a parameter is altered, keeping up with changes made when dragging or moving a point, line, or slider control. With a simple user interface, GeoGebra allows for constructions to be made through a menu of options, such as move, point, line, perpendicular line, polygon, circle, ellipse, angle, reflection, slider control, and move screen, among others.

We employ GeoGebra to build dynamic experiences with the geometric theorems stating the solution of the concurrency problems studied. The dynamic experiences were constructed using GeoGebra pages, which can be directly accessed from the text and interactively manipulated through the application. By adjusting parameters on the GeoGebra page, readers can assess the hypotheses and theses of each theorem that has been selected. For example, Figure 2 demonstrates two different positions of triangle ABC, achieved by relocating its vertices.

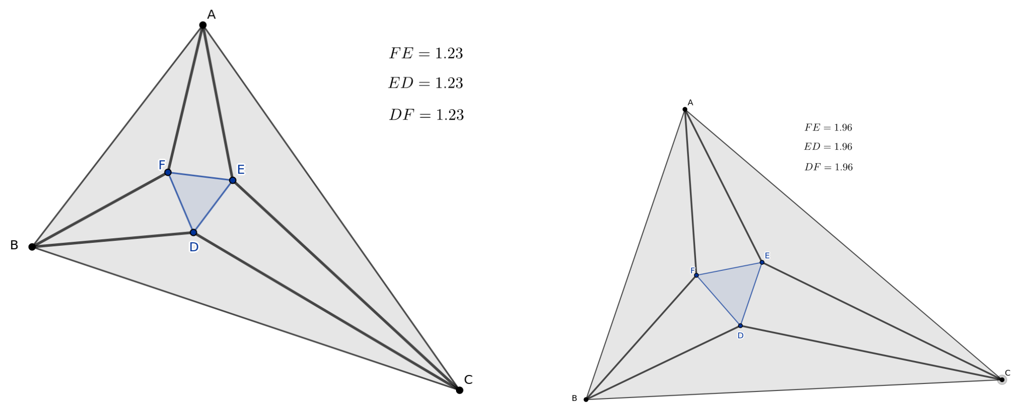


Figure 2: Dynamic movement of the vertices of triangle ABC.

Each link in the dynamic approaches has a title, an active file that can be modified, and a text at the bottom that cites the theorem and informs the parameters that can be changed. Thus, the reader can check that when all the hypotheses are met, the thesis is verified; when one of the hypotheses is not met, one of the parameters changes, and thus the theorem is invalid. This case is visually evident as the constructions no longer illustrate the proposed thesis.

Related Work: Several studies in the literature support the use of GeoGebra for showing geometric properties [15, 17, 18, 19]. For example, Nós, Sano, and Lago [14] employed GeoGebra to establish properties related to challenging geometric problems [16]. Lago and Nós [12] used the application to create dynamic animations illustrating geometric loci. Botana [7] harnessed GeoGebra for an interactive approach to generate geometric loci, and Gómez-Chacón and Escribano [9, 10] researched using this software in teaching geometric loci to pre-service teachers.

The spirit of our study is similar to the works of Alexander Bogomolny (1948-2018), an American mathematician born in Russia. Bogomolny's works, which are available on the internet and some of them we have used as references [2, 3, 4, 5, 6], have formal proofs in the body of the text and an external manipulable link where the reader can dynamically move some points and understand how the parameters are modified or how the thesis is proven true through the presented construction. Like Bogomolny, we offer the formal proof of the geometric theorem and then guide the reader to a dynamic experience in which the hypotheses can be modified and the visible results of these changes observed.

2 Miquel's point

Theorem 2.1, or Miquel's theorem, was proposed by a French mathematician, Auguste Miquel (1816-1851). This theorem establishes Miquel's point, a geometric locus defined by the intersection of three circumferences that pass through the vertices of a triangle. Miquel's theorem can be generalized for n circumferences [21], and there is a three-dimensional version for the two-dimensional case with $n = 3$ [11].

Theorem 2.1 (Miquel's theorem)**.** *Let ABC be a triangle. If A', B' and C' are points marked on the sides BC, AC, and AB, respectively (or on their extensions), then the three circles through any vertex of triangle ABC and the two marked points on the adjacent sides intersect at one point M.*

Proof. Let γ and Γ be the two circles through the points A, B', C' and B, A', C', respectively. They intersect at the point C' and at another point we call M; see Figure 3.

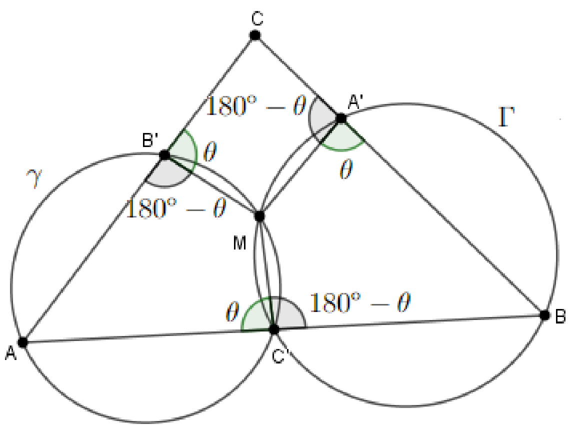


Figure 3: Miquel's M point.

Let $\angle AC'M = \theta$, so $\angle BC'M = 180° - \theta$. The angles $\angle AC'M$ and $\angle AB'M$ and the angles $\angle BC'M$ and $\angle BA'M$ are supplementary because the quadrilaterals $AC'MB'$ and $BC'MA'$ are inscribed in γ and Γ, respectively. Then, we have that $\angle AB'M = 180° - \theta$ and $\angle BA'M = \theta$. Thus $\angle CB'M = \theta$ and $\angle CA'M = 180° - \theta$.

Hence the angles $\angle CB'M$ and $\angle CA'M$ are supplementary, which shows that there is a circle Λ passing through the vertices of the quadrilateral $CB'MA'$. Therefore the three circles γ, Γ, and Λ intersect at point M. □

Point M from Theorem 2.1 is called Miquel's point.

Remark 2.2. The proof of Theorem 2.1 considers only the case where the points A', B', and C' are marked on the sides of triangle ABC.

We constructed the dynamic visualization of Miquel's theorem in GeoGebra and made it available at the link

https://www.geogebra.org/m/gbhdzk6t.

We can move points A, B, C, A', B', and C' dynamically in the GeoGebra App.

3 Fermat's points

The first Fermat point (or simply Fermat point or Torricelli point) arises from a problem proposed by the French mathematician Pierre de Fermat (1601-1665) to the Italian mathematician and physicist Evangelista Torricelli (1608-1647), inventor of the barometer. In a letter, Fermat challenged Torricelli to find a solution to the minimization problem whose resolution is the result of Theorem 3.1 below. Torricelli presented several solutions to the problem proposed by Fermat. In one of these solutions, he observed that the circumferences circumscribed to the equilateral triangles constructed externally on the sides of the triangle are concurrent at a point. We present the proof of Theorem 3.1 based on Napoleon's theorem [2, 18].

Theorem 3.1. *There exists a point such that the sum of its distances to the vertices of a triangle is minimal.*

Proof. Let ABC be a triangle. Consider a point F inside the triangle ABC and the triangle $A'BF'$ obtained from rotation by $60°$ of triangle ABF around B, as shown in Figure 4.

Thus $FA \equiv A'F'$. By construction, the triangle BFF' is equilateral, and so $FB \equiv F'F$. Then we have

$$FA + FB + FC \equiv A'F' + F'F + FC. \tag{3.1}$$

Since point A' is obtained by rotation of A around B, the position of A' does not depend on F. Furthermore, we have

$$FA + FB + FC \geq A'C.$$

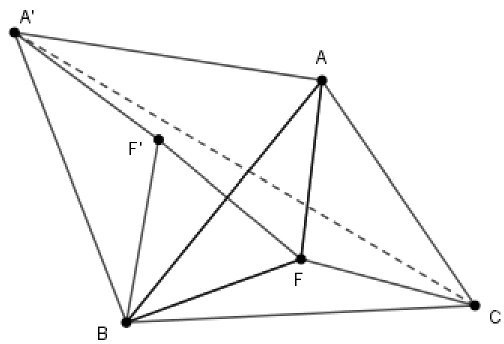


Figure 4: Triangles ABC, ABF, and $A'BF'$.

Hence the sum (3.1) is minimal if and only if $F \in A'C$. For that F, we have $\angle BFA' = 60°$ (if the rotation were made around point A, we would have $\angle AFA' = 60°$).

Now, constructing equilateral triangles on the sides of triangle ABC, we obtain the triangles ABA', ACC' and BCB' and the lines $A'C$, AB', and BC', see Figure 5.

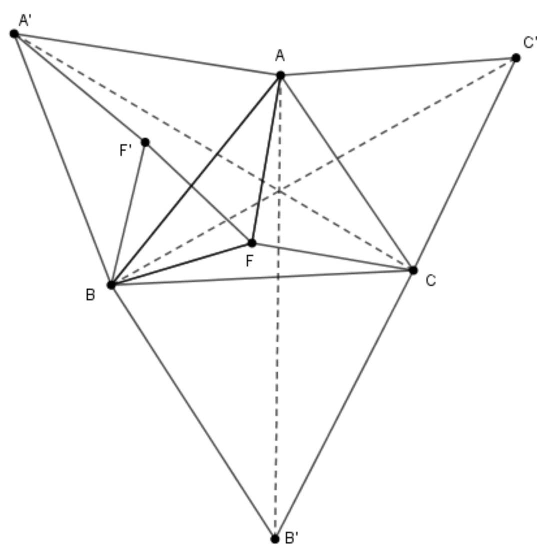


Figure 5: Fermat's point: the intersection between the lines AB', BC', and CA'.

By a similar reasoning to the one given earlier, we have that the point F that minimizes the sum $FA + FB + FC$ also lies in AB' and BC'. Napoleon's theorem guarantees that the three lines $A'C$, AB' and BC' intersect at point F, the angle between them is equal to 120° and $AB' \equiv BC' \equiv A'C = FA + FB + FC$. Therefore, point F is unique and lies in the intersection of the lines passing through the vertices of triangle ABC and the opposite vertices of equilateral triangles constructed on the sides of triangle ABC. In case one of the angles of the triangle ABC is greater than or equal to $120°$, point F will be the vertex corresponding to the greater triangle angle. □

Point F from Theorem 3.1 is called Fermat's point.

We constructed the dynamic visualization of Fermat's point in GeoGebra and made it available at the link

https://www.geogebra.org/m/smhageyu.

We can move points A, B, C, and M dynamically in the GeoGebra App.

3.1 Lighthouse at Fermat's points

Now that the Fermat point has been defined, one can further study its properties. A stunning property is given in Theorem 3.2 below, which provides a relationship between the Fermat point and the collinearity of points on the sides (or their extensions) of a triangle - as shown in Figure 6. Bogomolny [3] refers to this theorem as "a lighthouse at Fermat points".

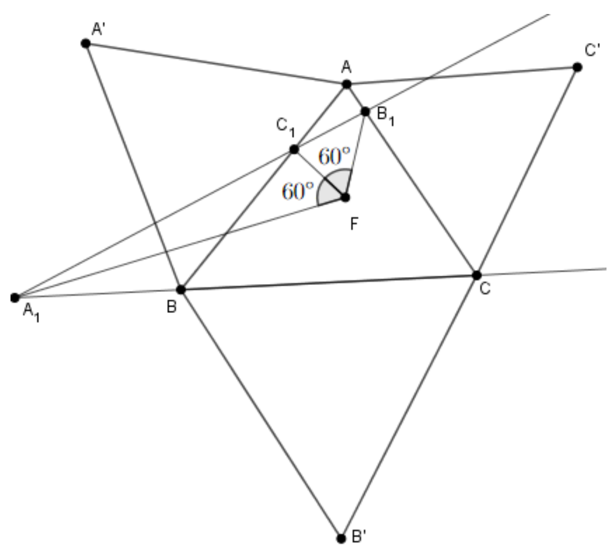


Figure 6: Lighthouse at Fermat's F point.

Theorem 3.2. *Let ABC be a triangle, F its Fermat point, and A_1, B_1, and C_1 are points in the lines BC, AC, and AB, respectively, such that $\angle A_1FC_1 \equiv \angle B_1FC_1 = 60°$. Then the points A_1, B_1, and C_1 are collinear.*

Proof. Let ABC be a triangle, the equilateral triangles $AA'B$, $BB'C$, and $CC'A$ constructed respectively on the side AB, BC, and AC of triangle ABC and the Fermat point $F \in AB' \cap A'C \cap BC'$. Let DEB be the homothetic image of the equilateral triangle BBC with homothetic center at point A. Thus $D \in AB$, $E \in AB'$, $B_1 \in AC$, $DB_1 \parallel BC$, $EB_1 \parallel B'C$ and $DE \parallel BB'$ as shown Figure 7.

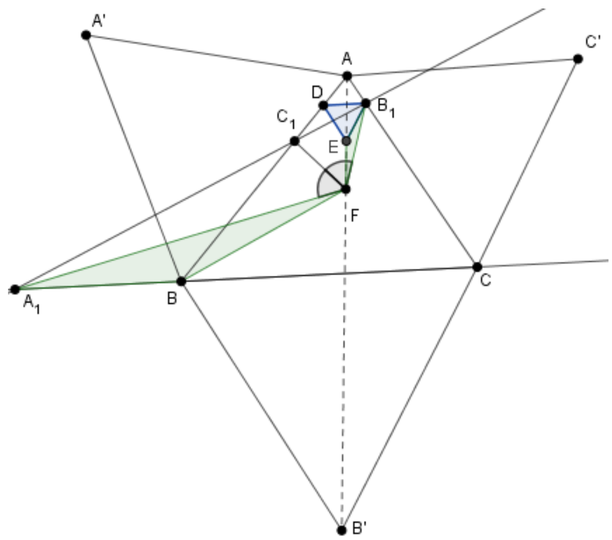


Figure 7: Triangles ABC and DEB_1, the Fermat point F, and points A_1, B_1, and C_1.

Let us denote by P the point of intersection of A_1B_1 and AB. We will show that $P \equiv C_1$, so A_1, B_1, and C_1 are collinear.

Given that $DB_1 \parallel BA_1$, the triangles B_1DP and A_1BP are similar. Hence

$$\frac{B_1D}{A_1B} = \frac{B_1P}{A_1P}. \tag{3.2}$$

Since the triangle DEB_1 is equilateral and $DB_1 \parallel BA_1$, it follows that

$$\angle(\overrightarrow{B_1E}, \overrightarrow{A_1B}) \equiv \angle(\vec{E}F, \vec{B}F) = 120°.$$

So we have that $\angle FEB_1 \equiv \angle FBA_1$. By construction, the angle $\angle B_1FA_1$ is equal to $120°$, and by the property of Fermat point, the angle $\angle AFB$ is equal to $120°$. This way, $\angle A_1FB \equiv \angle B_1FE$ and triangles B_1EF and A_1BF are similar. Thus we get that

$$\frac{B_1E}{A_1B} = \frac{FB_1}{FA_1}. \tag{3.3}$$

Using the relations (3.2) and (3.3) and the fact that $B_1D \equiv B_1E$, we obtain

$$\frac{B_1P}{A_1P} = \frac{FB_1}{FA_1}. \tag{3.4}$$

From (3.4), we conclude by the internal bisector theorem that FP is the bisector of the angle $\angle B_1FA_1$. Therefore $P \equiv C_1$. □

We constructed the dynamic visualization of the lighthouse at Fermat points in GeoGebra and made it available at the link
https://www.geogebra.org/m/ractsk7u.
We can move points A, B, C, and B_1 dynamically in the GeoGebra App.

4 Isotomic cevians

A cevian is a line segment that connects one of the vertices of a triangle to a point on the opposite side or its extension. The point of intersection between a cevian and the side or its extension of the triangle is known as the foot of the cevian. When the feet of two cevians, originating from the same vertex of a triangle, exhibit symmetry concerning the midpoint of the side opposite the shared vertex, these cevians are referred to as isotomic. Figure 8 illustrates the AA_1 and AA_2 cevians, whose feet are symmetric about the midpoint M of the side BC, opposite the angle $\angle BAC$.

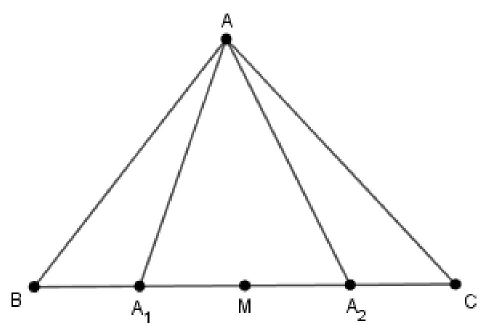


Figure 8: Isotomic cevians AA_1 and AA_2.

Let now AA_1, BB_1, and CC_1 be cevians of triangle ABC, with A_1, B_1, and C_1 denoting the respective footpoints of these cevians, and M_A, M_B, and M_C representing the midpoints of the sides BC, AC, and AB, correspondingly. We can establish that A_2, B_2, and C_2 are the symmetric points relative to A_1, B_1, and C_1 concerning MA,

MB, and MC. Consequently, the cevians AA_2, BB_2, and CC_2 are isotomic with the cevians AA_1, BB_1, and CC_1. Suppose the cevians AA_1, BB_1, and CC_1 intersect at a point P, and their isotomic cevians AA_2, BB_2, and CC_2 intersect at a point P'. In that case, these two points of intersection are termed isotomic conjugate points, as depicted in Figure 9.

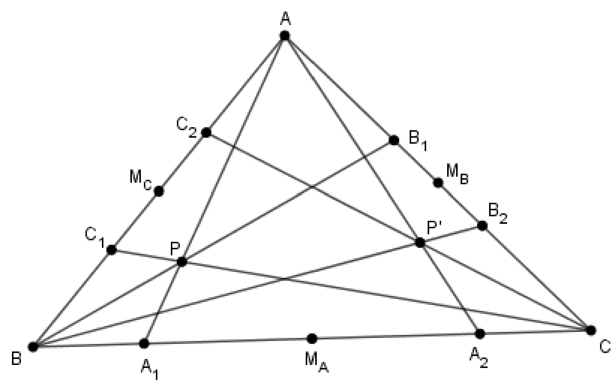


Figure 9: Isotomic conjugate points P and P'.

From the isotomic cevians AA_1 and AA_2, BB_1 and BB_2, and CC_1 and CC_2, it follows that

$$A_1B = -A_2C, \tag{4.1}$$
$$A_2B = -A_1C, \tag{4.2}$$
$$B_1C = -B_2A, \tag{4.3}$$
$$B_2C = -B_1A, \tag{4.4}$$
$$C_1A = -C_2B, \tag{4.5}$$
$$C_2A = -C_1B. \tag{4.6}$$

We present the proof of Theorem 4.1, which establishes a property related to isotomic cevians. The proof is based on Sortais and Sortais [20].

Theorem 4.1. *Isotomic cevians to three concurrent or parallel cevians are also concurrent or parallel.*

Proof. Let be the triangle ABC, the cevians AA_1, BB_1, and CC_1 with $A_1 \in BC$, $B_1 \in AC$, and $C_1 \in AB$, the points A_2, B_2, and C_2 symmetrical to the points A_1, B_1, and C_1 concerning M_A, M_B and M_C midpoints of segments BC, AC and AB, respectively, see Figure 9.

1. Suppose that the cevians AA_1, BB_1, and CC_1 are parallel. Let us show that $A_1 \notin \{B, C\}$. It is assumed, for instance, that $A_1 = B$. We have that $AA_1 \parallel AB$. Since $CC_1 \parallel AA_1$, it follows that $CC_1 \parallel AB$. So $CC_1 \cap AB = \emptyset$ given that $C \notin AB$. However, this contradicts the hypothesis because $CC_1 \cap AB = \{C_1\}$.

   Likewise, we can show that $B_1 \notin \{A, C\}$ and $C_1 \notin \{A, B\}$.

   As the cevians AA_1, BB_1, and CC_1 are parallel, we have by Ceva's theorem that

$$\frac{A_1B}{A_1C} \cdot \frac{B_1C}{B_1A} \cdot \frac{C_1A}{C_1B} = -1. \tag{4.7}$$

   Substituting (4.1)-(4.6) in (4.7) we get that

$$\frac{A_2C}{A_2B} \cdot \frac{B_2A}{B_2C} \cdot \frac{C_2B}{C_2A} = -1. \tag{4.8}$$

   From (4.8), it follows by Ceva's theorem that the cevians AA_2, BB_2, and CC_2, isotomic to the cevians AA_1, BB_1, and CC_1, respectively, are concurrent or parallel.

2. Suppose the cevians AA_1, BB_1, and CC_1 are concurrent at point P_1. Thus we have three possibilities to analyze.

   a) $P_1 \notin (AB \cup BC \cup CA)$.
   In this case, the same reasoning used previously allows deducing relation (4.8) from relation (4.7). Therefore, the AA_2, BB_2, and CC_2 cevians are concurrent or parallel.

   b) $P_1 \in (AB \cup BC \cup CA) - \{A, B, C\}$.
   Suppose, for example, that $P_1 \in AB - \{A, B\}$. We then have

$$\begin{aligned} AA_1 \equiv AP_1 &\Rightarrow A_1 = B; \\ BB_1 \equiv BP_1 &\Rightarrow B_1 = A; \\ CC_1 \equiv CP_1 &\Rightarrow P_1 = C_1. \end{aligned}$$

   The relations (4.1)-(4.6) guarantee that $A_2 \equiv C$ and $B_2 \equiv C$. Furthermore, C_2 is symmetric to P_1. So $AA_2 \cap BB_2 \cap CC_2 = AC \cap BC \cap CC_2 = \{C\}$.

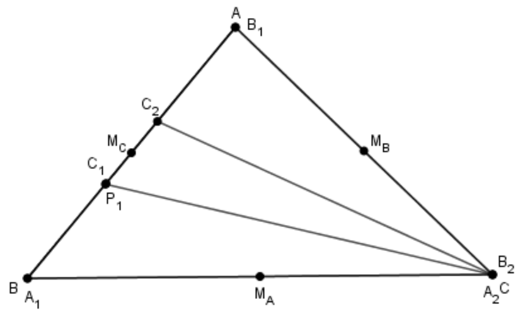


Figure 10: Triangle ABC with $A \equiv B_1$, $B \equiv A_1$, $C \equiv A_2 \equiv B_2$, and $P_1 \equiv C_1$.

Thus, the cevians AA_2, BB_2, and CC_2, isotomic to the cevians AA_1, BB_1, and CC_1, respectively, are concurrent in C, see Figure 10.

Likewise, if $P_1 \in BC - \{B, C\}$, then the cevians AA_2, BB_2, and CC_2 are concurrent in A, and if $P_1 \in AC - \{A, C\}$, then the cevians AA_2, BB_2, and CC_2 are concurrent in B.

c) $P_1 \in \{A, B, C\}$.

Suppose, for example, that $P_1 \equiv A$. Hence

$$BB_1 \equiv BP_1 \equiv BA \Rightarrow B_1 \equiv A;$$
$$CC_1 \equiv CP_1 \equiv CA \Rightarrow C_1 \equiv A.$$

In this way, the cevians AA_1, BB_1, and CC_1 are concurrent in A, see Figure 11.

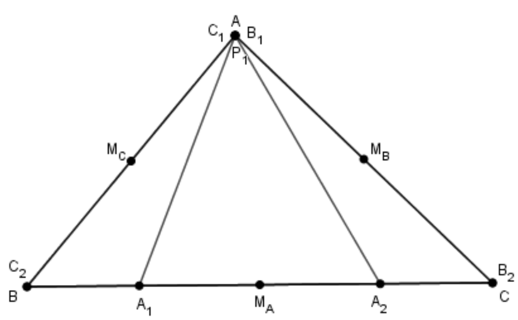


Figure 11: Triangle ABC with $A \equiv P_1 \equiv B_1 \equiv C_1$, $B \equiv C_2$, and $C \equiv B_2$.

So $AA_1 \not\equiv AB$ and $AA_1 \not\equiv AC$. Thereby $A_1 \in BC - \{B, C\}$ and

$$B_1 \equiv A \Rightarrow B_2 \equiv C;$$
$$C_1 \equiv A \Rightarrow C_2 \equiv B.$$

Since A_2 is symmetric to A_1 and $A_1 \in BC - \{B, C\}$, we get that $A_2 \in BC - \{B, C\}$. Hence $BB_2 \cap CC_2 \cap AA_2 = BC \cap BC \cap AA_2 = \{A_2\}$. Therefore, the cevians AA_2, BB_2, and CC_2 have a common point $A_2 \in BC - \{B, C\}$. That is, they are concurrent.

We obtain a similar result by assuming that $P_1 \equiv B$ or $P_1 \equiv C$.

□

We constructed the dynamic visualization of isotomic cevians in GeoGebra and made it available at the link

https://www.geogebra.org/m/mh42fvgy.

We can move points A, B, C, A_2, B_1, and C_2 dynamically in the GeoGebra App.

5 Gergonne's point

The locus in a triangle defined by the point of intersection of the cevians where the feet are the tangency points of the incircle of the triangle is called the Gergonne point, named in honor of the French mathematician Joseph Diez Gergonne (1771-1859).

Definition 5.1 (Gergonne's point)**.** Let Γ be the incircle of the triangle ABC and the points H, I, and J of the intersection of Γ with the sides BC, AC, and AB of triangle ABC, respectively. Point G, intersection of the AH, BI, and CJ segments, is the Gergonne point.

Figure 12 illustrates the Gergonne point. The proof of Theorem 5.2 follows Sortais and Sortais [20].

Theorem 5.2. *The Gergonne point exists and is unique.*

Proof. Let the triangle ABC circumscribed to the circle Γ and the points H, I, and J at which Γ touches the sides BC, AC, and AB of triangle ABC, respectively; see Figure 12.

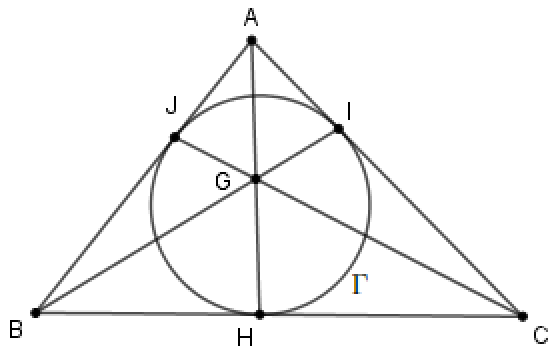


Figure 12: Gergonne's G point.

First, we will prove that the segments AH and BI are not parallel. Suppose that $AH \parallel BI$. By Thales' theorem, we have

$$\frac{HB}{HC} = \frac{AI}{AC}. \tag{5.1}$$

But $H \in BC - \{B, C\}$ and $I \in AC - \{A, C\}$. Thus

$$\frac{HB}{HC} < 0 \quad \text{and} \quad \frac{AI}{AC} > 0. \tag{5.2}$$

However, inequalities (5.2) contradict equality (5.1). Therefore the lines AH and BI are concurrent.

Now, we will show that the segments AH, BI, and CJ are concurrent at a point.

Since the circle Γ is inscribed in the triangle ABC, we have that the sides of ABC are perpendicular to the radius of Γ at the points of tangency, see Figure 13.

Let O be the center of Γ. Hence $OI \equiv OJ$ and triangles AOI and AOJ are congruent. From this, it follows that $AJ \equiv AI$. Analogously, $BH \equiv BJ$ and $CH \equiv CI$.

Considering $AJ = x$, $BH = y$, and $CH = z$, we get

$$\frac{BH}{CH} \cdot \frac{CI}{AI} \cdot \frac{AJ}{BJ} = \left(-\frac{y}{z}\right) \cdot \left(-\frac{z}{x}\right) \cdot \left(-\frac{x}{y}\right) = -1. \tag{5.3}$$

From (5.3) and Ceva's theorem, it follows that the lines containing the segments AH, BI, and CJ are concurrent or parallel. As we have already shown that they are not parallel, the lines are concurrent at the Gergonne point.

□

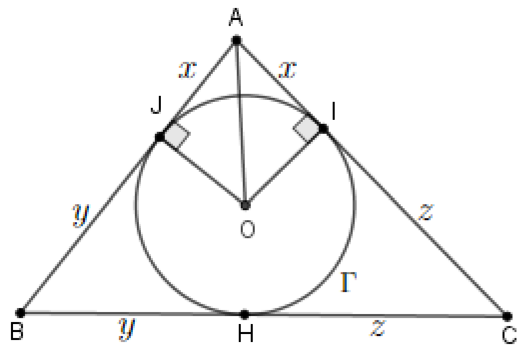


Figure 13: Triangle ABC, inscribed circumference Γ, and intersection points H, I, and J.

We constructed the dynamic visualization of the Gergonne point in GeoGebra and made it available at the link

https://www.geogebra.org/m/r2euvfx9.

We can move points A, B, and C dynamically in the GeoGebra App.

6 Nagel's point

The locus in a triangle defined by the point of intersection of the cevians that join the vertices of the triangle to the points of tangency of the excircles on the opposite sides is called the Nagel point, named in honor of the German geometer Christian Heinrich von Nagel (1803-1882).

Definition 6.1 (Nagel's point)**.** Let Γ_A, Γ_B, and Γ_C be the excircles of the triangle ABC and D, E, and F the points of tangency of these circles to the sides BC, AC, and AB of triangle ABC, respectively. Cevians AD, BE, and CF intersects at a single point called Nagel's point.

Figure 14 illustrates Gergonne and Nagel points.

The Gergonne and Nagel points are examples of isotomic conjugate points. The following result - Theorem 6.2, whose proof can be found in Neto [13], will be needed to demonstrate the existence and uniqueness of the Nagel point.

Theorem 6.2. *If ABC is any triangle, point Q is the incenter of ABC, point I_B is the excenter of ABC relative to the side AC, and M is the point of intersection between the circle that circumscribes ABC and the segment QI_B, then M is the midpoint of the arc AC which does not contain B and $MA \equiv MC \equiv MQ \equiv MI_B$.*

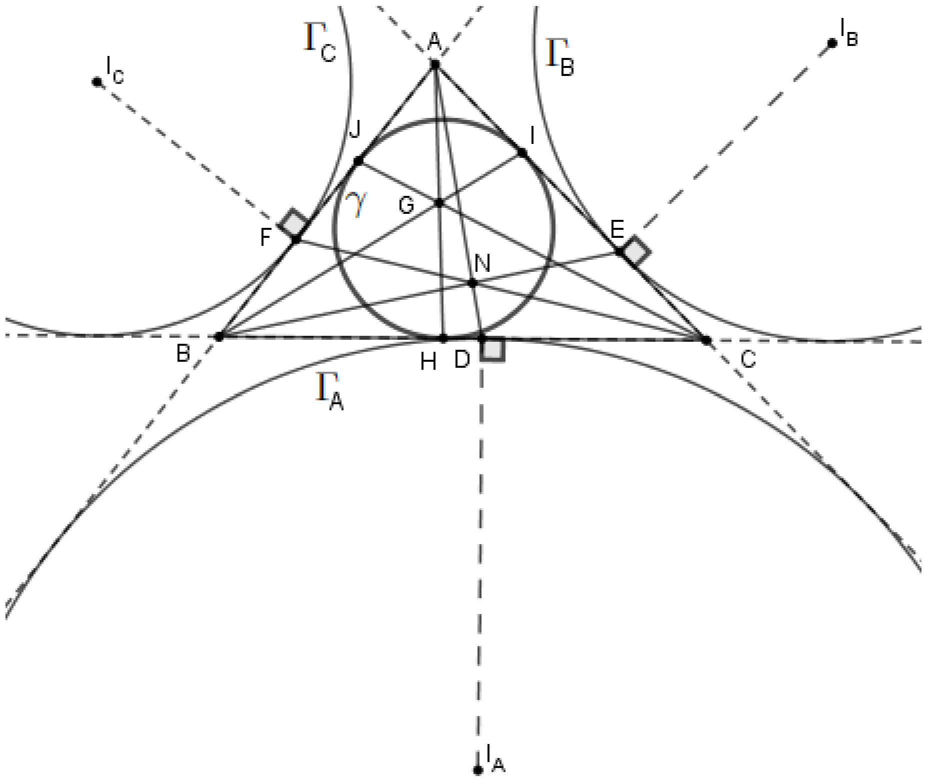


Figure 14: Isotomic conjugate points: the Gergonne point G and the Nagel point N.

Theorem 6.3. *The Nagel point exists and is unique.*

Proof. Let ABC be a triangle, γ the inscribed circle with center Q, Γ the circumscribed circle with center O, Γ_B the excircle relative to the side AC with center I_B, S the midpoint of AC, the point $I = \gamma \cap AC$, the point $E = \Gamma_B \cap AC$, the point $M = \Gamma \cap QI_B$ and the bisector r of AC, see Figure 15.

By Theorem 6.2, we have that $MA \equiv MC \equiv MQ \equiv MI_B$. The points Q, A, I_B, and C are vertices of an inscriptible quadrilateral of diameter QI_B, whose midpoint M is the center of the circumscribed circle. This property is valid if triangle ABC is not isosceles with base AC. If triangle ABC is isosceles with base AC, as shown in Figure 16, then the points Q, I_B, S, and O are colinear, and the line that passes through them coincides with the bisector of AC. In this case, the center of the circumcircle of quadrilateral $AQCI_B$ is also the midpoint M of the segment QI_B.

Thus, we have that the orthogonal projections of the points Q, I_B, and M on the segment AC are, respectively, the points I, E, and S. Since M is the midpoint of QI_B,

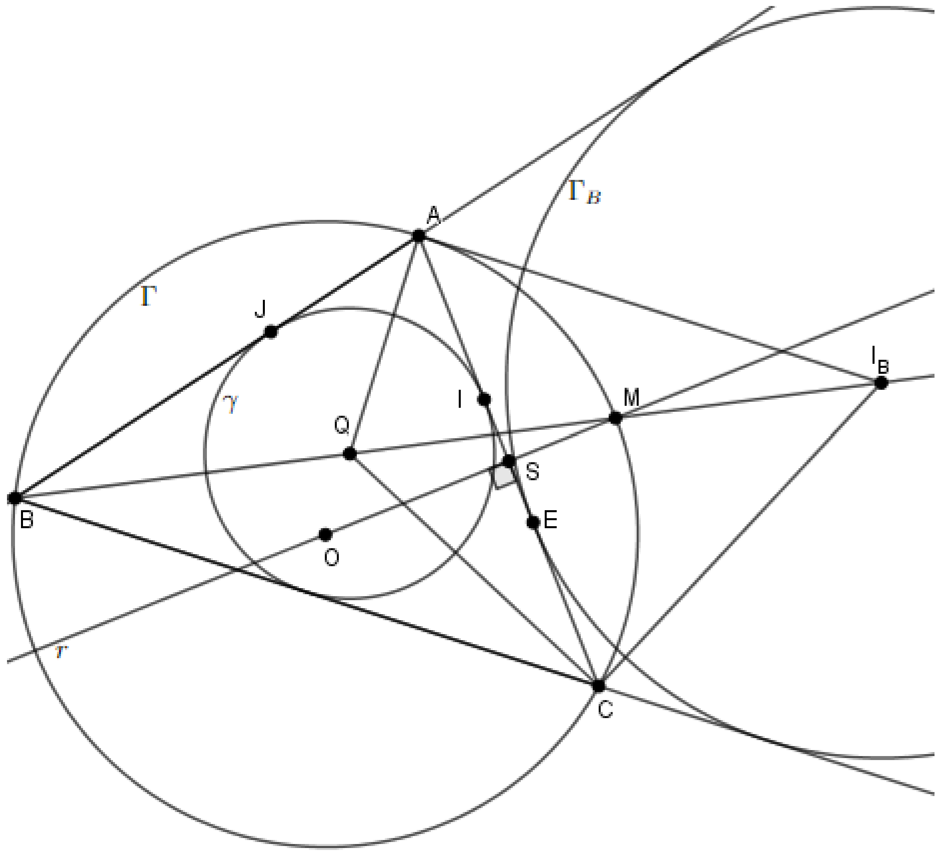


Figure 15: Triangle ABC and three circumferences: inscribed γ, circumscribed Γ, and ex-inscribed Γ_B regarding the side AC.

then the points I and E are symmetric concerning the point S. This holds for the three ex-inscribed circles and the three sides of triangle ABC.

In this way, we have in Figure 14 that D and H, I and E, and J and F are isotomic conjugate points. Hence, AD, BE, and CF are, respectively, the isotomic cevians of AH, BI, and CJ, which concur at point N. This point is interior to triangle ABC since $H \in BC$ and $I \in AC$. Therefore, point N exists and is unique.

□

We constructed the dynamic visualization of the Nagel point in GeoGebra and made it available at the link

https://www.geogebra.org/m/zqnbanpd.

We can move points A, B, and C dynamically in the GeoGebra App.

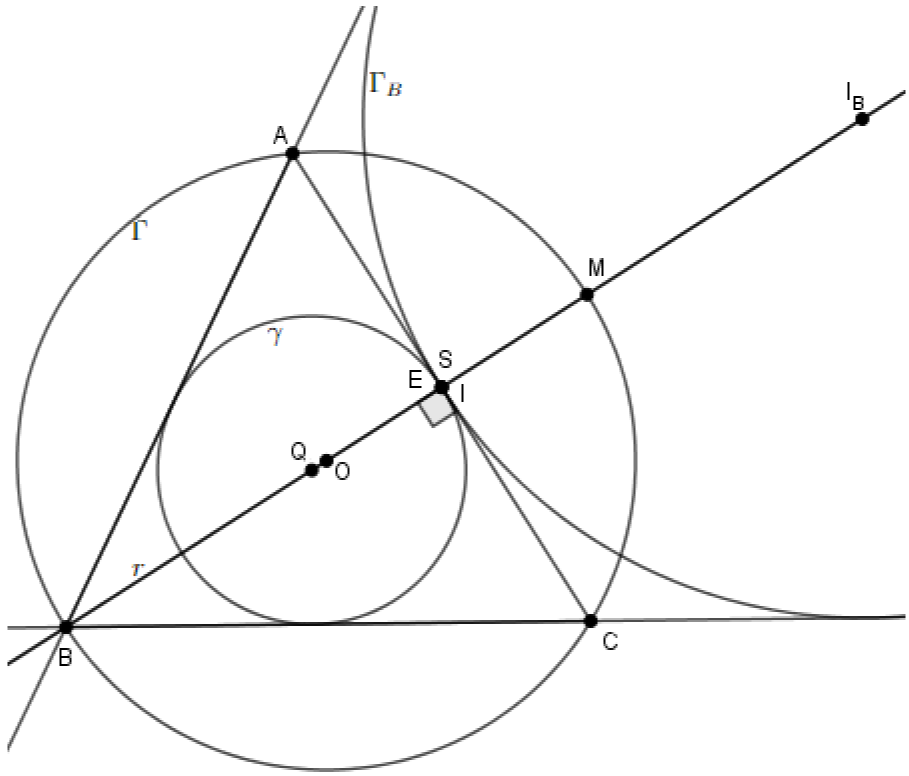


Figure 16: Isosceles triangle ABC and three circumferences: inscribed γ, circumscribed Γ, and ex-inscribed Γ_B regarding the side AC.

7 Concluding remarks

In this study, we have provided proofs and dynamic visualizations of geometric loci, such as Miquel, Fermat, Gergonne, and Nagel points. Throughout our investigations, we made use of the freely available application GeoGebra.

We aimed to create a resource for students and educators in mathematics-teaching degree programs, offering active and engaging strategies for understanding theorems associated with geometric loci. A possible future development along the lines of the present work can be the inclusion of descriptive guides in the dynamic approaches.

References

[1] Arcavi, Abraham. The role of visual representations in the learning of mathematics. Educational Studies in Mathematics **52(3)** (2003), 215-241.

[2] Bogomolny, Alexander. *The Fermat point and generalizations*, 2018a. http://www.cut-the-knot.org/Generalization/fermat_point.shtml. Last checked on Aug 09, 2023.

[3] Bogomolny, Alexander. *Lighthouse at Fermat points*, 2018b. https://www.cut-the-knot.org/m/Geometry/2x60.shtml. Last checked on Aug 09, 2023.

[4] Bogomolny, Alexander. *Miquel's point: what is it? a mathematical droodle*, 2018c. https://www.cut-the-knot.org/Curriculum/Geometry/Miquel.shtml. Last checked on Aug 09,2023.

[5] Bogomolny, Alexander. *Nagel point: what is it about? a mathematical droodle*, 2018d. http://www.cut-the-knot.org/Curriculum/Geometry/NagelPoint.shtml#%20explanation. Last checked on Aug 09, 2023.

[6] Bogomolny, Alexander. *Nobbs' points, Gergonne line: what are they? a mathematical droodle*. https://www.cut-the-knot.org/Curriculum/Geometry/Nobbs.shtml. Last checked on Aug 09, 2023.

[7] Botana, Francisco. Interactive versus symbolic approaches to plane loci generation in dynamic geometry environments. Lectures Notes in Computer Science **2330** (2002), 211-218.

[8] GeoGebra. *GeoGebra for teaching and learning math*, 2023. https://www.geogebra.org/?lang=en. Last checked on Aug 08, 2023.

[9] Gómez-Chacón, Inés M.; Escribano, Jesús. Geometric locus activities in a dynamic geometry system: non-iconic visualization and instrumental genesis. Revista Latinoamericana de Investigación en Matemática Educativa **17(4-2)** (2014), 361-376.

[10] Gómez-Chacón, Inés M.; Escribano, Jesús. Teaching geometric locus using GeoGebra an experience with pre-service teachers, 2011. https://www.semanticscholar.org/paper/Teaching-Geometric-Locus-using-GeoGebra-An-with-G%C3%B3mez-Chac%C3%B3n-Escribano/737b97b276862a7f962d3f29e300b0f91af2fa29. Last checked on Aug 08, 2023.

[11] Handwiki. *Miquel's theorem*, 2022. https://handwiki.org/wiki/Miquel%27s_theorem. Last checked on Aug 09, 2023.

[12] Lago, Rodrigo C.; Nós, Rudimar L. Investigating plane geometry theorems with GeoGebra. Revista do Instituto GeoGebra Internacional de São Paulo **9(3)** (2020), 15-29.

[13] Neto, Antonio C. M. *Geometria*. SBM, Rio de Janeiro, 2013.

[14] Nós, Rudimar L.; Sano, Mari; Lago, Rodrigo C. Using GeoGebra in generalization processes of geometrical challenging problems. Revista do Instituto GeoGebra Internacional de São Paulo **10(1)** (2021), 49-64.

[15] Nós, Rudimar L.; Sano, Mari; Tavares, Maria C. F. P. Using Bernoulli numbers to generalize a limit of finite sum arising from volume computations with the squeeze theorem. Revista Sergipana de Matemática e Educação Matemática **6(3)** (2021), 77-96.

[16] Posamentier, Alfred S.; Salkind, Charles T. *Challenging problems in geometry*. Dover, New York, 1996.

[17] Silva, Vitoria M. R. da; Nós, Rudimar L.; Sano, Mari. A dynamic view of the pythagorean theorem with GeoGebra. Revista do Instituto GeoGebra Internacional de São Paulo **12(1)** (2023), 62-77.

[18] Silva, Victoria M. R. da. *A dynamic view of some classical geometric theorems via GeoGebra*. Masters dissertation, Universidade Tecnológica Federal do Paraná, Curitiba, Paraná, 2022.

[19] Silva, Victoria M. R. da; Nós, Rudimar L. *Calculando o volume de poliedros convexos*. CRV, Curitiba, 2018.

[20] Sortais, Ivonne; Sortais, René. *La géométrie du triangle: exercices résolus*. Hermann, Paris, 1997.

[21] Weisstein, Eric W. *Miquel's theorem*, 2023. https://mathworld.wolfram.com/MiquelsTheorem.html. Last checked on Aug 09, 2023.

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