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THE GOLDMAN-TUCKER THEOREM FOR TWO-SIDE CONSTRAINTS LINEAR OPTIMIZATION PROBLEM

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Resumo

O Teorema de Goldman-Tucker detém uma importância significativa na otimização linear, pois garante a existência de uma solução estritamente complementar. Duas razões destacam a sua relevância: em primeiro lugar, os métodos de pontos interiores com barreira logarítmica convergem para uma solução estritamente complementar, cuja existência é garantida por este teorema. Em segundo lugar, o Teorema de Goldman-Tucker, juntamente com a condição de complementariedade do sistema KKT (Karush–Kuhn–Tucker), motivou o desenvolvimento de precondicionadores eficientes para as iterações finais dos métodos de pontos interiores, como o precondicionador Separador. Enquanto textos acadêmicos sobre otimização linear geralmente apresentam este resultado e suas consequências apenas para o problema de otimização linear na forma canônica ou forma padrão, o objetivo deste artigo é elucidar de maneira acurada o teorema para o problema de otimização linear canalizado, acompanhado por uma prova detalhada. Além disso, é apresentado um resultado teórico que utiliza este teorema para estabelecer a convergência do trajetória central para uma solução estritamente complementar, juntamente com um exemplo ilustrativo de aplicação de ambos resultados teóricos.

Palavras-chave: Método de pontos interiores; Trajetória central; Solução estritamente complementar.

Abstract

The Goldman-Tucker theorem holds significant importance in linear optimization, as it ensures the existence of a strictly complementary solution. Two reasons underscore its relevance: firstly, logarithmic barrier interior point methods converge towards a strictly complementary solution, whose existence is warranted by this theorem. Secondly, the Goldman-Tucker theorem, coupled with the complementary slackness condition of the KKT (Karush–Kuhn–Tucker) system, has motived the development of efficient preconditioners for the final iterations of interior point methods, such as the Splitting preconditioner. Academic texts of linear optimization present this result as well as its consequences only for the canonical and standard form. This paper aims to accurately elucidate this theorem for the two-side constraints linear optimization problem with a detailed demonstration. Additionally, a theoretical result that uses this theorem to show that the central path converges to a strictly complementary solution and an example of application of both theoretical results are presented.

Keywords: Interior point method; Central path; Strictly complementary solution.

1 Introduction

In linear optimization, first-order conditions, also known as the Karush-Kuhn-Tucker conditions, are necessary and sufficient for achieving optimality [4]. Among these, the complementarity condition stands out because the Interior Point method with the logarithmic barrier uses its own perturbation to propose a rather efficient algorithm. Moreover, this condition motivated the construction of preconditioners, such as the Splitting preconditioner [5].

The Goldman-Tucker theorem guarantees the existence of at least one strictly complementary optimal solution, see [2], [1] and [3]. When using the Interior Point method to solve a linear programming problem, it is possible to show that this method converges to a strictly complementary solution, see [3] [6].

Academic texts in the field typically describe and demonstrate the main results of linear optimization problems only in the context of canonical or standard forms. However, there exist problems with two-sided constraints, and few works explicitly describe their fundamental properties. Therefore, there is a need to present, demonstrate, and apply the Goldman-Tucker theorem to this type of linear programming problem.

This work is based on Chapter 2 of the book [7] and Chapter 3 of the book [6], where they deal with the duality theory and optimality conditions for the standard and canonical linear programming problem.

In Section 2, we will introduce the necessary definitions pertaining to the Goldman-Tucker theorem within the context of linear optimization problems with two-sided constraints. Moving on to Section 3, we provide a detailed proof of the main result of this paper, Theorem 3.1. In addition we demonstrate an application of the Goldman-Tucker theorem to establish the convergence of the central path towards a strictly complementary solution, as demonstrated in Theorem 3.3. In Example 1, we apply the results obtained from both Theorem 3.1 and Theorem 3.3. Finally, in Section 4, we present our conclusions.

2 Mathematical preliminaries

We establish some results about duality theory and optimality conditions for the two-side constrains linear optimization problem (P) and its dual problem (D):

$$\text{(P)}\begin{cases}\min & c^Tx\\ \textbf{s. t.} & Ax=b\\ & x+s=u\\ & (x,s)\geq 0\end{cases} \qquad \text{(D)}\begin{cases}\max & b^Ty-u^Tw\\ \textbf{s. t.} & A^Ty-w+z=c\\ & (w,z)\geq 0\\ & y\in\mathbb{R}^m.\end{cases}, \tag{2.1}$$

where $x, s, w, z \in \mathbb{R}^n$. We assume that $A \in \mathbb{R}^{m\times n}$ has full row rank and both (P) and (D) have solutions.

The sets of feasible points of (P) and (D) are:

$$\mathcal{F}(P) = \left\{(x,s)\in\mathbb{R}^{2n}_+ \mid Ax=b,\ x+s=u\right\} \quad \text{and}$$
$$\mathcal{F}(D) = \left\{(y,w,z) \mid y\in\mathbb{R}^m, (w,z)\in\mathbb{R}^{2n}_+,\ A^Ty-w+z=c\right\},$$

respectively.

Using the KKT conditions, (x^*, s^*) is the optimal solution of the problem (P) given in (2.1) if, and only if, exist $y^*\in\mathbb{R}^m$, z^*, $w^*\in\mathbb{R}^n_+$ that satisfy the following equations:

$$Ax = b, \quad x\geq 0; \tag{2.2}$$
$$x+s = u, \quad s\geq 0; \tag{2.3}$$
$$A^Ty-w+z = c, \quad w,z\geq 0; \tag{2.4}$$
$$XZe = 0; \tag{2.5}$$
$$WSe = 0, \tag{2.6}$$

where $e = (1,\ldots,1)\in\mathbb{R}^n$, X, Z, W and S are diagonal matrices defined as $X = diag(x)$, $Z = diag(z)$, $W = diag(w)$ and $S = diag(s)$, respectively.

The sets of solutions of (P) and (D) are defined as:

$$\mathcal{P}^* = \left\{(x^*, s^*) \in \mathcal{F}(P) | c^T x^* \leq c^T x, \text{ for all } (x, s) \in \mathcal{F}(P)\right\} \quad \text{and}$$
$$\mathcal{D}^* = \left\{(y^*, w^*, z^*) \in \mathcal{F}(D) | b^T y^* - u^T w^* \geq b^T y - u^T w, \text{ for } (y, w, z) \in \mathcal{F}(D)\right\},$$

respectively.

Equations (2.5) and (2.6) are called complementary slackness conditions. That is, every optimal solution $(x^*, s^*) \in \mathcal{P}^*$, $(y^*, w^*, z^*) \in \mathcal{D}^*$ satisfies $X^* Z^* e = 0$ and $S^* W^* e = 0$.

Definition 2.1 (Strict complementary slackness condition)**.** *The solutions* $(x^*, s^*) \in \mathcal{P}^*$ *and* $(y^*, w^*, z^*) \in \mathcal{D}^*$ *satisfy the strict complementary slackness condition when* $x^* + z^* > 0$ *and* $s^* + w^* > 0$*.*

From equations (2.5) and (2.6) of the KKT system, we know that

$$\begin{aligned} x_i^* = 0 \quad &\text{and/or} \quad z_i^* = 0 \quad \text{for all} \quad i = 1, 2, \ldots, n; \\ w_i^* = 0 \quad &\text{and/or} \quad s_i^* = 0 \quad \text{for} \quad i = 1, 2, \ldots, n, \end{aligned}$$

where $(x^*, s^*) \in \mathcal{P}^*$ and $(y^*, w^*, z^*) \in \mathcal{D}^*$.

The next definitions are used in the Goldman-Tucker result for the primal-dual pair of the two-side constrains linear optimization problem given by (2.1).

Definition 2.2.

- *For* $(x^*, s^*) \in \mathcal{P}^*$*, we define* $\mathcal{B}_1(x^*) = \{i : x_i^* > 0\}$ *and*

$$\mathcal{B}_1 = \bigcup_{(x^*, s^*) \in \mathcal{P}^*} \mathcal{B}_1(x^*). \tag{2.7}$$

- *For* $(y^*, w^*, z^*) \in \mathcal{D}^*$*, we define* $\mathcal{N}_1(z^*) = \{i : z_i^* > 0\}$ *and*

$$\mathcal{N}_1 = \bigcup_{(y^*, w^*, z^*) \in \mathcal{D}^*} \mathcal{N}_1(z^*). \tag{2.8}$$

- *For* $(x^*, s^*) \in \mathcal{P}^*$ *we define* $\mathcal{B}_2(s^*) = \{i : s_i^* > 0\}$*, and*

$$\mathcal{B}_2 = \bigcup_{(x^*, s^*) \in P^*} \mathcal{B}_2(s^*). \tag{2.9}$$

- *For* $(y^*, w^*, z^*) \in D^*$*, we define* $\mathcal{N}_2(w^*) = \{i : w_i^* > 0\}$*, and*

$$\mathcal{N}_2 = \bigcup_{(y^*,w^*,z^*)\in D^*} \mathcal{N}_2(w^*). \tag{2.10}$$

The next result, Theorem 3.1, is the Goldman-Tucker theorem for the primal-dual pair given by (2.1). The proof is based on Theorem 3.3.5 (Strict Complementarity Theorem for the Standard pair) of the book [6] increasing the notations of sets: $\mathcal{B}_1$, $\mathcal{B}_2$, $\mathcal{N}_1$ and $\mathcal{N}_2$, given by the Definition 2.2 in order to clarify all the details on the proof of this important theorem.

3 Main Results

3.1 The Goldman-Tucker Theorem for two-side constraints linear optimization problem

Theorem 3.1. *Consider the sets established in Definition 2.2, then:*

$$\mathcal{B}_i \cup \mathcal{N}_i = \{1, 2, \ldots, n\} \quad \text{and} \quad \mathcal{B}_i \cap \mathcal{N}_i = \emptyset \quad \text{for} \quad i = 1, 2. \tag{3.1}$$

Consequently, there are $(x^*, s^*) \in \mathcal{P}^*$ *and* $(y^*, w^*, z^*) \in \mathcal{D}^*$ *such that:* $x^* + z^* > 0$ *and* $s^* + w^* > 0$*.*

Proof. For $i = 1, 2$, we define the sets: $\overline{\mathcal{N}}_i = \{1, 2, \ldots, n\} \backslash \mathcal{B}_i$. We use this definition in order to prove: $\overline{\mathcal{N}}_i = \mathcal{N}_i$.

- Case $i = 1$

  Let $i \in \mathcal{N}_1$, this implies that there is $z^* \in \mathbb{R}^n_+$ such that $(y^*, w^*, z^*) \in \mathcal{D}^*$ with $z_i^* > 0$. Using the complementary slackness conditions given by (2.5), $x_i^* = 0$ for all $x^* \in \mathbb{R}^n_+$ with $(x^*, s^*) \in \mathcal{P}^*$. Thus, $i \notin \mathcal{B}_1$, this means that $i \in \overline{\mathcal{N}}_1$. Therefore, $\mathcal{N}_1 \subseteq \overline{\mathcal{N}}_1$.

  In order to prove $\overline{\mathcal{N}}_1 \subseteq \mathcal{N}_1$, we prove the existence of $(y^*, w^*, z^*) \in \mathcal{D}^*$ with $z^*_{\overline{\mathcal{N}}_1} > 0$, consequently $\overline{\mathcal{N}}_1 = \mathcal{N}_1(z^*) \subseteq \mathcal{N}_1$.

  Let "k" be the optimal value of the primal-dual pair given in (2.1). We define the vector $\overline{e}^T = (0_{\mathcal{B}_1}, 1_{\bar{\mathcal{N}}_1})^T$ and considering the following primal-dual pair of linear programming problem:

$$(\mathrm{P}') \begin{cases} \min & -\bar{e}^T x \\ \textbf{s. t.} & Ax = b \\ & -c^T x - t = -k \\ & x + s = u \\ & (x, s, t) \geq 0 \end{cases} \qquad (\mathrm{D}') \begin{cases} \max & b^T y - kt - u^T w \\ \textbf{s. t.} & A^T y - tc - w + z = -\bar{e} \\ & (w, z, t) \geq 0 \\ & y \in \mathbb{R}^m. \end{cases} \tag{3.2}$$

We observe that (x, s, t) is a feasible point of (P') if, and only if, $(x, s) \in \mathcal{P}^*$. In fact, $(x, s, t) \in \mathcal{F}(\mathrm{P}')$ if, and only if, $Ax = b$, $x + s = u$ and $(x, s) \geq 0$. Then, $(x, s) \in \mathcal{F}(P)$ and therefore $c^T x \geq k$. Moreover, $c^T x \leq c^T x + t = k$, so $c^T x = k$. Thus, $(x, s) \in \mathcal{P}^*$. Now, $(x, s) \in \mathcal{P}^*$ implies that $(x, s, t) \in \mathcal{F}(\mathrm{P}')$, it is true trivially.

We observe that the optimal value of the problems (3.2) is 0. In fact, let $(x, s, t) \in \mathcal{F}(\mathrm{P}')$, then $(x, s) \in \mathcal{P}^*$. Note that $i \in \overline{\mathcal{N}}_1$ if, and only if, $i \notin \mathcal{B}_1$, if, and only if, $x_i^* = 0$ for all x^* with $(x^*, s^*) \in \mathcal{P}^*$. Hence, the objective function of (P') is zero for every feasible point. Consequently, the optimal value of (P') and (D') is zero.

Let $(\bar{y}, \bar{w}, \bar{z}) \in \mathcal{D}^*$ and $(\hat{y}, \hat{t}, \hat{w}, \hat{z}) \in (\mathcal{D}')^*$, these points are used in order to define:

$$y^* = \frac{\bar{y} + \hat{y}}{1 + \hat{t}}, \quad w^* = \frac{\bar{w} + \hat{w}}{1 + \hat{t}} \quad \text{and} \quad z^* = \frac{\bar{z} + \hat{z} + \bar{e}}{1 + \hat{t}}. \tag{3.3}$$

Next, we prove that (y^*, w^*, z^*) defined by (3.3) is a solution of the problem (D) given in (2.1). Moreover, $z^*_{\overline{\mathcal{N}}_1} > 0$. In fact, from $(\bar{y}, \bar{w}, \bar{z}) \in \mathcal{F}(D)$, $(\hat{y}, \hat{t}, \hat{w}, \hat{z}) \in \mathcal{F}(D')$ and (3.3), we have that $w^* \geq 0$, $z^* \geq 0$ and $z_{\overline{\mathcal{N}}_1} > 0$. Now, we prove that $(y^*, w^*, z^*) \in \mathcal{F}(D)$:

$$\begin{aligned} A^T y^* - w^* + z^* &= \frac{1}{1 + \hat{t}} \left(A^T \bar{y} + A^T \hat{y} - \bar{w} - \hat{w} + \bar{e} + \bar{z} + \hat{z} \right) \\ &= \frac{1}{1 + \hat{t}} \left(A^T \bar{y} - \bar{w} + \bar{z} \right) + \frac{1}{1 + \hat{t}} \left(A^T \hat{y} - \hat{w} + \hat{z} - c\hat{t} \right) + \\ &\quad + \frac{1}{1 + \hat{t}} \left(\bar{e} + c\hat{t} \right) \\ &= \frac{1}{1 + \hat{t}} c - \frac{1}{1 + \hat{t}} \bar{e} + \frac{1}{1 + \hat{t}} \left(\bar{e} + c\hat{t} \right) \\ &= c \frac{1 + \hat{t}}{1 + \hat{t}} = c\,. \end{aligned}$$

Moreover, observe that,

$$\begin{aligned} b^T y^* - u^T w^* =& b^T \left(\frac{\bar{y}+\hat{y}}{1+\hat{t}}\right) - u^T \left(\frac{\bar{w}+\hat{w}}{1+\hat{t}}\right) \\ =& \frac{1}{1+\hat{t}} \left(b^T\bar{y} - u^T\bar{w}\right) + \left((b^T\hat{y} - u^T\hat{w} - k\hat{t}) + k\hat{t}\right) \\ =& \frac{1}{1+\hat{t}} \left(k + (0 + k\hat{t})\right) = \ k\,. \end{aligned}$$

Hence, (y^*, w^*, z^*) given by (3.3) is a solution for the problem (D) with $z^*_{\overline{\mathcal{N}}_1} > 0$. Then, $\overline{\mathcal{N}}_1 = \mathcal{N}_1(z^*) \subseteq \mathcal{N}_1$ and therefore, $\mathcal{N}_1 = \overline{\mathcal{N}}_1$.

- Case $i = 2$

  Let $i \in \mathcal{N}_2$, this implies that there is $w^* \in \mathbb{R}^n_+$ such that $(y^*, w^*, z^*) \in \mathcal{D}^*$ with $w^*_i > 0$. By the complementary slackness condition in the equation (2.6), $s^*_i = 0$ for every $s^* \in \mathbb{R}^n_+$ with $(x^*, s^*) \in \mathcal{P}^*$. Thus $i \notin \mathcal{B}_2$, or equivalently, $i \in \overline{\mathcal{N}}_2$. Therefore, $\mathcal{N}_2 \subseteq \overline{\mathcal{N}}_2$.

  In order to verify the other inclusion, we prove the existence of the vector $(y^*, w^*, z^*) \in \mathcal{D}^*$ such that $w_{\overline{\mathcal{N}}_2} > 0$. Hence $\overline{\mathcal{N}}_2 = \mathcal{N}_2(w^*) \subseteq \mathcal{N}_2$.

  Let "k" be the optimal value of primal-dual pair given by (2.1). We define the vector $\hat{e}^T = (0_{\mathcal{B}_2}, 1_{\bar{\mathcal{N}}_2})^T$, considering the following primal-dual pair of linear programming problem:

$$(\mathrm{P}'')\begin{cases} \min & \hat{e}^T s \\ \textbf{s. t.} & Ax = b \\ & -c^T x - t = -k \\ & x + s = u \\ & (x, s, t) \geq 0 \end{cases} \qquad (\mathrm{D}'')\begin{cases} \max & b^T y - kt - u^T w \\ \textbf{s. t.} & A^T y - ct - w + z = 0 \\ & (z, t) \geq 0 \\ & w \geq \hat{e} \\ & y \in \mathbb{R}^m. \end{cases} \tag{3.4}$$

  We note that (x, s, t) is a feasible point of (P'') if, and only if, $(x, s) \in \mathcal{P}^*$. In fact, $(x, s, t) \in \mathcal{F}(\mathrm{P}'')$ if, and only if, $Ax = b$, $x + s = u$ and $(x, s) \geq 0$. Thus, $(x, s) \in \mathcal{F}(P)$ and $c^T x \geq k$. In addition, $c^T x \leq c^T x + t = k$, so $c^T x = k$, consequently, $(x, s) \in \mathcal{P}^*$. Now, $(x, s) \in \mathcal{P}^*$ implies that $(x, s, t) \in \mathcal{F}(\mathrm{P}'')$, it is true trivially.

  Now, we will prove that the optimal value of the problems (3.4) is zero. Let $(x, s, t) \in \mathcal{F}(\mathrm{P}')$, this is, $(x, s) \in \mathcal{P}^*$. Observe $i \in \overline{\mathcal{N}}_2$ if, and only if, $i \notin \mathcal{B}_2$ if, and only if, $s^*_i = 0$ for all s^* with $(x^*, s^*) \in \mathcal{P}^*$. Hence, the objective function of (P'')

is zero for every feasible point. Consequently, the optimal value of (P$''$) and (D$''$) is zero.

Let $(\bar{y}, \bar{w}, \bar{z}) \in \mathcal{D}^*$ and $(\hat{y}, \hat{t}, \hat{w}, \hat{z}) \in (D'')^*$, these points are used in order to define:

$$y^* = \frac{\bar{y} + \hat{y}}{1 + \hat{t}}, \quad w^* = \frac{\bar{w} + \hat{w}}{1 + \hat{t}} \quad \text{and} \quad z^* = \frac{\bar{z} + \hat{z}}{1 + \hat{t}}. \tag{3.5}$$

Next, we prove that the point (y^*, w^*, z^*) defined by (3.5) is a solution of (D) with $w^*_{\bar{N}_2} > 0$. In fact, from $(\bar{y}, \bar{w}, \bar{z}) \in \mathcal{F}(D)$, $(\hat{y}, \hat{t}, \hat{w}, \hat{z}) \in \mathcal{F}(D'')$ and $\hat{w} \geq \hat{e}$, we have $w^* \geq 0$ and $z^* \geq 0$. In particular, $w^*_{\overline{\mathcal{N}}_2} > 0$. Note that:

$$\begin{aligned}
A^T y^* - w^* + z^* &= \frac{1}{1 + \hat{t}} \left(A^T \bar{y} + A^T \hat{y} - \bar{w} - \hat{w} + \bar{z} + \hat{z} \right) \\
&= \frac{1}{1 + \hat{t}} \left(A^T \bar{y} - \bar{w} + \bar{z} \right) + \frac{1}{1 + \hat{t}} \left(A^T \hat{y} - \hat{w} + \hat{z} - c\hat{t} \right) + \\
&\quad + \frac{1}{1 + \hat{t}}\, c\hat{t} \\
&= \frac{1}{1 + \hat{t}} c - \frac{1}{1 + \hat{t}} 0 + \frac{1}{1 + \hat{t}} c\hat{t} = c\, \frac{1 + \hat{t}}{1 + \hat{t}} = c\,,
\end{aligned}$$

then, $(y^*, w^*, z^*) \in \mathcal{F}(D)$.

In addition, .

$$\begin{aligned}
b^T y^* - u^T w^* &= b^T \left(\frac{\bar{y} + \hat{y}}{1 + \hat{t}} \right) - u^T \left(\frac{\bar{w} + \hat{w}}{1 + \hat{t}} \right) \\
&= \frac{1}{1 + \hat{t}} \left(b^T \bar{y} - u^T \bar{w} \right) + \left((b^T \hat{y} - u^T \hat{w} - k\hat{t}) + k\hat{t} \right) \\
&= \frac{1}{1 + \hat{t}} \left(k + (0 + k\hat{t}) \right) = k\,.
\end{aligned}$$

Hence, (y^*, w^*, z^*) defined by (3.5) is a solution of the dual problem (D), with $w^*_{\overline{\mathcal{N}}_2} > 0$. Then $\overline{\mathcal{N}}_2 = \mathcal{N}_2(w^*) \subseteq \mathcal{N}_2$ and $\mathcal{N}_2 = \overline{\mathcal{N}}_2$.

Therefore, we prove that $\mathcal{B}_i \cup \mathcal{N}_i = \{1, 2, \ldots, n\}$ for $i = 1, 2$.
In order to prove that $\mathcal{B}_i \cap \mathcal{N}_i = \emptyset$, two cases are also considered:

- Case $i = 1$

  If there happened to be an index j that belonged to $\mathcal{B}_1$ and $\mathcal{N}_1$, then, there is $x^* \in \mathbb{R}^n_+$ with $(x^*, s^*) \in \mathcal{P}^*$ such that $x^*_j > 0$, and there is $z^* \in \mathbb{R}^n_+$ with $(y^*, w^*, z^*) \in \mathcal{D}^*$ such that $z^*_j > 0$. Therefore, $x^*_j z^*_j > 0$ contradicting the complementary slackness condition (2.5). Thus $\mathcal{B}_1 \cap \mathcal{N}_1 = \emptyset$.

- Case $i = 2$

  If there happened to be an index j that belonged to $\mathcal{B}_2$ and $\mathcal{N}_2$, then, there is $s^* \in \mathbb{R}^n_+$ with $(x^*, s^*) \in \mathcal{P}^*$ such that $s^*_j > 0$, and there is $w^* \in \mathbb{R}^n_+$ with $(y^*, w^*, z^*) \in \mathcal{D}^*$ such that $w^*_j > 0$. Therefore, $s^*_j w^*_j > 0$ contradicting the complementary slackness condition (2.6). Thus $\mathcal{B}_2 \cap \mathcal{N}_2 = \emptyset$.

Thus, for $i = 1, 2$, $\mathcal{B}_i$ and $\mathcal{N}_i$ are a partition of the set $\{1, 2, \ldots, n\}$.

In order to prove the existence of a strictly complementary solution of the primal-dual pair given by (2.1), we consider an arbitrary primal solution $(x^*, s^*) \in \mathcal{P}^*$. If $x^*_i > 0$ for any $i \in \{1, 2, \ldots, n\}$ then $x^*_i + z^*_i > 0$ for all dual solution $(y^*, w^*, z^*) \in \mathcal{D}^*$. If $x^*_i = 0$, for all $i \in \{1, 2, \ldots, n\}$, we consider the following two cases:

- There is another primal solution $(\hat{x}^*, \hat{s}^*) \in \mathcal{P}^*$ such that $\hat{x}^*_i > 0$. Observe that the vector $(\bar{x}^*, \bar{s}^*) = t(x^*, s^*) + (1 - t)(\hat{x}^*, \hat{s}^*)$ is a convex combination of (x^*, s^*) and $(\hat{x}^*, \hat{s}^*)$ is also a solution of (P), moreover $\bar{x}^*_i > 0$. Hence $\bar{x}^*_i + z^*_i > 0$ for all $(y^*, w^*, z^*) \in \mathcal{D}^*$.

- There is no optimal primal solution $(\hat{x}^*, \hat{s}^*) \in \mathcal{P}^*$ such that $\hat{x}^*_i > 0$. That is, the index $i \notin \mathcal{B}_1$, then $i \in \mathcal{N}_1$, then there is a dual solution $(y^*, w^*, z^*) \in \mathcal{D}^*$ with $z^*_i > 0$. Thus, $x^*_i + z^*_i > 0$.

Similarly, we prove that there is an index $i \in \{1, 2, \ldots, n\}$ such that $s^*_i + w^*_i > 0$.

□

3.2 Central path converges to a strictly complementary solution

Primal-dual Interior point methods are closely related to the barrier methods developed by Fiacco and McCormick in 1960s. If the logarithmic barrier penalty is applied to the non-negativity restrictions of the primal problem (P) given in (2.1), we obtain:

$$\begin{cases} \min & c^T x - \mu \left(\sum_{i=1}^n \log x_i + \sum_{i=1}^n \log s_i \right) \\ \textbf{s. t.} & Ax = b\,, \\ & x + s = u\,, \\ & x_i, s_i > 0, \text{for } i = 1, 2 \ldots, n \end{cases} \tag{3.6}$$

for $i = 1, 2 \ldots, n$, where $\mu > 0$ is a positive parameter that controls the relationship between the barrier term and the objective function of the problem (P). The term μ is known as *duality measure*.

Because $\mu > 0$, (3.6) is a convex problem, then the KKT conditions are sufficient and necessary to find its solution. That is, $(x^*, s^*) = (x^*(\mu), s^*(\mu))$ is a solution of the problem (3.6) if, and only if, there are $y^* = y^*(\mu) \in \mathbb{R}^m$, $z^* = z^*(\mu)$ and $w^* = w^*(\mu) \in \mathbb{R}^n_{++}$ that satisfy the following equations:

$$Ax = b, \quad x > 0; \tag{3.7}$$
$$x + s = u, \quad s > 0; \tag{3.8}$$
$$A^T y - w + z = c, \quad w, z > 0; \tag{3.9}$$
$$XZe = \mu e; \tag{3.10}$$
$$WSe = \mu e. \tag{3.11}$$

Using the Implicit Function theorem, it is proved that for every $\mu > 0$ there is only one vector $(x(\mu), s(\mu), y(\mu), w(\mu), z(\mu))$ that satisfies the (3.7)-(3.11) equations. These points implicitly define a curve known as central path. This curve stabilizes primal-dual algorithms by providing a route that can be followed to the solution sets $\mathcal{P}^*$ and $\mathcal{D}^*$.

Using the equations (3.7)-(3.11), we define:

$$F : \mathbb{R} \times \mathbb{R}^n_{++} \times \mathbb{R}^n_{++} \times \mathbb{R}^m \times \mathbb{R}^n_{++} \times \mathbb{R}^n_{++} \to \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$$
$$\begin{pmatrix} \mu \\ x \\ s \\ y \\ w \\ z \end{pmatrix} \mapsto \begin{pmatrix} Ax - b \\ x + s - u \\ A^T y - w + z - c \\ XZe - \mu e \\ WSe - \mu e \end{pmatrix}.$$

Observe that for every $\hat{\mu} \in \mathbb{R}_{++}$ and $(x, s, y, w, z)^T \in \mathbb{R}^n_{++} \times \mathbb{R}^n_{++} \times \mathbb{R}^m \times \mathbb{R}^n_{++} \times \mathbb{R}^n_{++}$, the matrix given by (3.12):

$$\nabla F \begin{pmatrix} \hat{\mu} \\ x \\ s \\ y \\ w \\ z \end{pmatrix} = \begin{pmatrix} A & 0 & 0 & 0 & 0 \\ I_n & I_n & 0 & 0 & 0 \\ 0 & 0 & A^T & -I_n & I_n \\ Z & 0 & 0 & 0 & X \\ 0 & W & 0 & S & 0 \end{pmatrix} \tag{3.12}$$

is non singular. Thus, by the Implicit Function Theorem, there are a neighborhood U

of $\hat{\mu} > 0$, a neighborhood V of $(x, s, y, w, z)^T$ and a transformation

$$\varphi : U \to V \quad \text{given by}$$
$$\mu \mapsto \begin{pmatrix} x(\mu) \\ s(\mu) \\ y(\mu) \\ w(\mu) \\ z(\mu) \end{pmatrix}.$$

That is, φ is a parametrization of the central path. We formally define this trajectory by $\mathcal{C} = \{\varphi(\mu) : \mu > 0\}$.

In order to prove that the central path converges to a strictly complementary solution, it is necessary to establish the next result.

Lemma 3.2. *Let* $(x^*, s^*) \in \mathcal{P}^*$, $(y^*, w^*, z^*) \in \mathcal{D}^*$ *and*

$$\varphi(\mu) = (x(\mu), s(\mu), y(\mu), w(\mu), z(\mu))^T \in \mathcal{C}\,.$$

Then:

$$x^T(\mu)z^* + (x^*)^T z(\mu) + s^T(\mu)w^* + (s^*)^T w(\mu) = 2n\mu. \tag{3.13}$$

Proof. Observe that $x(\mu) - x^* \in Null(A)$ and $(z^* - z(\mu)) + (w(\mu) - w^*) \in Im(A^T)$. In fact, using (3.7) and (3.9), we have:

$$\begin{aligned} A(x(\mu) - x^*) &= Ax(\mu) - Ax^* = b - b = 0, \quad \text{and} \\ A^T(y(\mu) - y^*) &= A^T y(\mu) - A^T y^* \\ &= (c + w(\mu) - z(\mu)) - (c + w^* - z^*) \\ &= (z^* - z(\mu)) + (w(\mu) - w^*). \end{aligned}$$

From (3.10) and (3.11), we note that $x^T(\mu)z(\mu) = n\mu$ and $s^T(\mu)w(\mu) = n\mu$, respectively.

Thus, using (3.8), we have:

$$\begin{aligned} 0 &= (x(\mu)-x^*)^T((z^*-z(\mu))+(w(\mu)-w^*)) \\ &= x^T(\mu)z^* - x^T(\mu)z(\mu) + x^T(\mu)w(\mu) - x^T(\mu)w^* \\ &\quad - (x^*)^T z^* + (x^*)^T z(\mu) - (x^*)^T w(\mu) + (x^*)^T w^* \\ &= x^T(\mu)z^* - n\mu + (u-s(\mu))^T w(\mu) - x^T(\mu)w^* \\ &\quad + (x^*)^T z(\mu) - (u-s^*)^T w(\mu) + (u-s^*)^T w^* \\ &= x^T(\mu)z^* - n\mu + \cancel{u^T w(\mu)} - s^T(\mu)w(\mu) - x^T(\mu)w^* \\ &\quad + (x^*)^T z(\mu) - \cancel{u^T w(\mu)} + (s^*)^T w(\mu) + u^T w^* - (s^*)^T w^* \\ &= x^T(\mu)z^* - n\mu - n\mu + (u-x(\mu))^T w^* \\ &\quad + (x^*)^T z(\mu) + (s^*)^T w(\mu) \\ &= x^T(\mu)z^* + (x^*)^T z(\mu) - 2n\mu + s^T(\mu)w^* + (s^*)^T w(\mu)). \end{aligned}$$

Thus, we obtain the desired result. □

The next theorem is the second most important result of this work. It uses the Definition 2.2, Theorem 3.1 and Lemma 3.2, in order to prove that the central path converges to a strictly complementary solution.

Theorem 3.3. *If* $\lim_{\mu\to 0}\varphi(\mu) = (\bar{x}, \bar{s}, \bar{y}, \bar{w}, \bar{z})$, *where* φ *denotes the parametrization of the central path, then* $\mathcal{B}_1 = \mathcal{B}_1(\bar{x})$, $\mathcal{N}_1 = \mathcal{N}_1(\bar{z})$, $\mathcal{B}_2 = \mathcal{B}_2(\bar{s})$ *and* $\mathcal{N}_2 = \mathcal{N}_2(\bar{w})$, *where* $\mathcal{B}_1$, $\mathcal{N}_1$, $\mathcal{B}_2$ *and* $\mathcal{N}_2$ *are defined by* (2.7), (2.8), (2.9) *and* (2.10), *respectively. That is, the central path converges to a strictly complementary solution.*

Proof. Let $(x(\mu), s(\mu), y(\mu), w(\mu), z(\mu))^T \in \mathcal{C}$ for any $\mu > 0$. From (3.13), we have:

$$0 \le (x^*)^T z(\mu) \le 2n\mu,$$

further, since $x^* \ge 0$ and $z(\mu) > 0$, we note:

$$0 \le x_i^* z_i(\mu) \le 2n\mu, \quad \text{for} \quad i = 1, 2, \ldots, n. \tag{3.14}$$

Substituting $z_i(\mu) = \mu x_i^{-1}(\mu)$ in the equation (3.14), we have:

$$0 \le x_i^* \le 2n x_i(\mu), \quad \text{for all} \quad i = 1, 2, \ldots, n. \tag{3.15}$$

If $\mu \to 0$, then,

$$0 \le x_i^* \le 2n\bar{x}_i, \quad \text{for every} \quad i = 1, 2, \ldots, n. \tag{3.16}$$

From (3.16), we note that,

$$\bar{x}_i = 0 \quad \text{implies that} \quad x_i^* = 0, \quad \text{for all} \quad x^* \in \mathcal{P}^*. \tag{3.17}$$

Similarly, using (3.13) it is possible to prove that:

$$\bar{s}_i = 0 \quad \text{implies that} \quad s_i^* = 0, \quad \text{for} \quad s^* \in \mathcal{P}^*, \tag{3.18}$$

$$\bar{z}_i = 0 \quad \text{implies that} \quad z_i^* = 0, \quad \text{for all} \quad z^* \in \mathcal{D}^*, \tag{3.19}$$

$$\bar{w}_i = 0 \quad \text{implies that} \quad w_i^* = 0, \quad \text{for every} \quad w^* \in \mathcal{D}^*. \tag{3.20}$$

Since $\lim_{\mu\to 0}\varphi(\mu) = (\bar{x}, \bar{s}, \bar{y}, \bar{w}, \bar{z})$ and φ is the implicit parameterization given by equations (3.7)-(3.11), we have $(\bar{x}, \bar{s}, \bar{y}, \bar{w}, \bar{z})$ is solution of (P)-(D) because satisfies the KKT conditions, the equations (2.2)-(2.6).

In order to prove that $\mathcal{B}_1 = \mathcal{B}_1(\bar{x})$, we note that $\mathcal{B}_1(\bar{x}) \subseteq \mathcal{B}_1$ because $\bar{x}$ is a solution of the problem (P). Now, consider $i \in \mathcal{B}_1$, then there is $x^* \in \mathcal{P}^*$ such that $x_i^* > 0$. Using (3.17), we conclude that $\bar{x}_i > 0$. Thus, $i \in \mathcal{B}_1(\bar{x})$, consequently $\mathcal{B}_1 \subseteq \mathcal{B}_1(\bar{x})$ and $\mathcal{B}_1 = \mathcal{B}_1(\bar{x})$.

In a similar way, we use (3.18),(3.19) and (3.20) in order to prove $\mathcal{B}_2 = \mathcal{B}_2(\bar{s})$, $\mathcal{N}_1 = \mathcal{N}_1(\bar{z})$ and $\mathcal{N}_2 = \mathcal{N}_2(\bar{w})$, respectively. □

The following example uses the theorems presented in this text.

3.3 A two-side constraints linear optimization problem to apply the theorems of this paper

We use the following problem to apply the results of theorems 3.1 and 3.3.

Example 1. *Consider the two-side constraints linear optimization problem.*

$$\begin{cases} \min & x_1 - x_2 + x_3 \\ \textbf{s. t.} & x_1 + x_2 + x_3 = 1 \\ & x_i + s_i = 0.9, \quad i = 1,2,3 \\ & (x_i, s_i) \geq 0, \quad i = 1,2,3. \end{cases} \tag{3.21}$$

The problem (3.21) has infinite solutions given by $x^* = ((1-\lambda)0.1, \quad 0.9, \quad 0.1\lambda)^T$, $s^* = (0.8 + 0.1\lambda, \quad 0, \quad 0.9 - 0.1\lambda)^T$, where $\lambda \in [0,1]$. The sets specified in Definition 2.2 in this example are:

- If $\lambda = 0$, $\mathcal{B}_1(x^*) = \{1,2\}$ and $\mathcal{B}_2(s^*) = \{1,3\}$.

- When $0 < \lambda < 1$, $\mathcal{B}_1(x^*) = \{1,2,3\}$ and $\mathcal{B}_2(s^*) = \{1,3\}$.
- If $\lambda = 1$, $\mathcal{B}_1(x^*) = \{2,3\}$ and $\mathcal{B}_2(s^*) = \{1,3\}$.

Therefore,

$$\mathcal{B}_1 = \bigcup_{x^* \in \mathcal{P}^*} \mathcal{B}_1(x^*) = \{1,2,3\} \quad \text{and} \quad \mathcal{B}_2 = \bigcup_{s^* \in \mathcal{P}^*} \mathcal{B}_2(s^*) = \{1,3\}. \tag{3.22}$$

The dual problem of (3.21) is:

$$\begin{cases} \max & y - 0.9w_1 - 0.9w_2 - 0.9w_3 \\ \textbf{s. t.} & y + z_1 - w_1 = 1 \\ & y + z_2 - w_2 = -1 \\ & y + z_3 - w_3 = 1 \\ & (z_i, w_i) \geq 0, \quad \text{for } i = 1,2,3. \end{cases} \tag{3.23}$$

Problem given by (3.23) has only one solution:

$$y = 1, \quad z^* = (0,0,0)^T \quad \text{and} w^* = (0,2,0)^T,$$

then: $\mathcal{N}_1(z^*) = \emptyset$, $\mathcal{N}_2(w^*) = \{2\}$. Therefore:

$$\mathcal{N}_1 = \bigcup_{z^* \in \mathcal{D}^*} \mathcal{N}_1(z^*) = \emptyset \quad \text{and} \quad \mathcal{N}_2 = \bigcup_{w^* \in \mathcal{D}^*} \mathcal{N}_2(w^*) = \{2\}. \tag{3.24}$$

Now, we sketch the central path as accurately as possible, see Figure 1. The next system of equations defines implicitly the parameterization of the central path in this example:

$$\begin{cases} x_1 + x_2 + x_3 = 1 \\ x_i + s_i = u_i \quad i = 1,2,3 \\ y + z_i - w_i = c_i \quad \text{for every } i = 1,2,3 \\ x_i z_i = \mu, \quad \text{for } i = 1,2,3 \\ s_i w_i = \mu, \quad \text{for all } i = 1,2,3 \\ (x_i, s_i, z_i, w_i) \geq 0, \quad \text{for every } i = 1,2,3. \end{cases} \tag{3.25}$$

In order to simplify notation, we use $(x, s, y, z, w) \in \mathcal{C}$ instead

$$(x(\mu), s(\mu), y(\mu), z(\mu), w(\mu)) \in \mathcal{C}.$$

For $i = 1, 2, 3$, we have:

$$0 \leq z_i = c_i - y + w_i \quad \text{or} \quad y \leq c_i + w_i.$$

Since $x_i z_i = \mu$, we obtain:

$$x_i = \frac{\mu}{c_i - y + w_i} \quad \text{and} \tag{3.26}$$

from $x_i + s_i = u_i$ and $s_i w_i = \mu$:

$$x_i = \frac{u_i w_i - \mu}{w_i}\,. \tag{3.27}$$

Using (3.26) and (3.27), we have:

$$u_i w_i^2 + (u_i(c_i - y) - 2\mu)w_i - (u_i - y)\mu = 0\,. \tag{3.28}$$

The solution of (3.28) is

$$w_i = \frac{-(u_i(c_i - y) - 2\mu) \pm ((u_i(c_i - y))^2 + 4\mu^2)^{1/2}}{2u_i}\,. \tag{3.29}$$

In this example, $c_1 = c_3$ and $u_1 = u_2 = u_3$. From (3.29), we conclude that $w_1 = w_3$ and $s_1 = s_3$. In addition, using (3.27), $x_1 = x_3$ and $z_1 = z_3$.

Now, from (3.29), we obtain:

$$c_i - y + w_i = \frac{u_i(c_i - y) + 2\mu \pm ((u_i(c_i - y))^2 + 4\mu^2)^{1/2}}{2u_i}\,. \tag{3.30}$$

and we can substitute (3.30) in (3.26) and then (3.26) in $x_1 + x_2 + x_3 = 1$ to obtain:

$$\begin{aligned} &2(c_2 - y)(2\mu - \left((u_1(c_1 - y))^2 + 4\mu^2\right)^{1/2} + \\ &\quad + (c_1 - y)(2\mu - \left((u_1(c_2 - y))^2 + 4\mu^2\right)^{1/2} = 2(1 - 3/2u_1)(c_1 - y)(c_2 - y). \end{aligned} \tag{3.31}$$

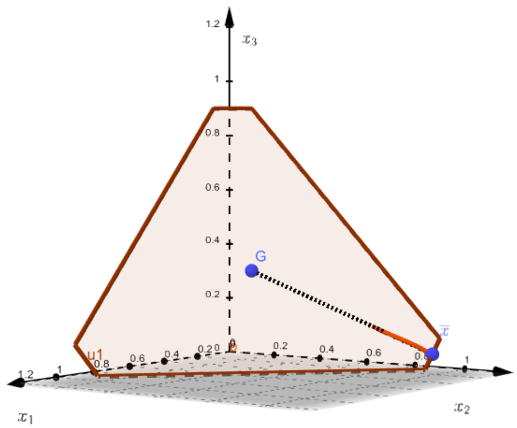
Note that (3.31) defines implicitly y and μ.

Using Bézier curve, parametric curves that approximates y and μ are given by:

$$\tilde{\mu}(t) = 2 \cdot 0.05t(1 - t) + 0.32t^2; \tag{3.32}$$

$$\tilde{y}(t) = (1 - t)^2 + 2(-0.28)t(1 - t) - 0.22t^2, \tag{3.33}$$

Figure 1: Approximation for primal central path



for $t \in [0,1]$. In this way, using (3.32) in (3.33), it is possible to obtain a function that approximates y such that it depends on μ in the interval $[0, 0.2]$:

$$\tilde{y}(\tilde{\mu}) = \left(1-\tilde{t}\right)^2 + 2(-0.28)\tilde{t}(1-\tilde{t}) - 0.22\tilde{t}^2 \tag{3.34}$$

where $\tilde{t} = t(\tilde{\mu})$ is an inverse function of $\tilde{\mu}$.

Finally, using (3.34), (3.29) and (3.26), the Figure 1 shows the parametric functions $(\tilde{x}_1(\tilde{\mu}), \tilde{x}_2(\tilde{\mu}), \tilde{x}_3(\tilde{\mu}))$ for $\tilde{\mu}$ in $[0, 0.2]$. Where $\tilde{x}_i$ approximates primal central path.

We note that the parametric functions $(\tilde{x}_1(\tilde{\mu}), \tilde{x}_2(\tilde{\mu}), \tilde{x}_3(\tilde{\mu}))$ for values of μ near zero converges to $(0.05, 0.9, 0.05)^T$. In fact, the central path of the primal-dual pair given in (3.21)-(3.23) converges to:

$$\bar{x} = \begin{pmatrix} 0.05 \\ 0.9 \\ 0.05 \end{pmatrix}, \ \bar{s} = \begin{pmatrix} 0.85 \\ 0 \\ 0.85 \end{pmatrix}, \ \bar{y} = 1, \ \bar{z} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \bar{w} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}. \tag{3.35}$$

Further, we highlight two important observations: The solution given by (3.35) is a strictly complementary solution and, from (3.22), (3.24) and (3.35), we check that: $\mathcal{B}_1 = \mathcal{B}_1(\bar{x})$, $\mathcal{B}_2 = \mathcal{B}_2(\bar{s})$, $\mathcal{N}_1 = \mathcal{N}_1(\bar{z})$ and $\mathcal{N}_2 = \mathcal{N}_2(\bar{w})$.

4 Conclusions

The intention of this paper is to contribute with well-dated academic material that allows a better understanding of the Goldman-Tucker theorem for the two-side constraints linear optimization problem, see Theorem 3.1. The motivation is the fact that this result is scarcely studied for this formulation in academic materials in the area. We presented some definitions and results of duality theory and optimality conditions for this type of optimization problem.

In addition, in order to present an application of this theorem: a mathematical result which proves that the central path converges to a strictly complementary solution, see the Theorem 3.3. Finally, in order to illustrate the concepts and theorems studied in this paper, we present the Example 1, which uses the Bézier curve in order to present an approximation of a parametrization of the central path that converges to a strictly complementary solution.

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