REVISEM

Revista Sergipana de Matemática e Educação Matemática

https://doi.org/10.34179/revisem.v9i2.19821

HIGH-ORDER ZIG-ZAG THEORIES TO STATIONARY LAMINATED COMPOSITE BEAM ANALYSIS BY THE RAYLEIGH-RITZ METHOD

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Abstract

Technological advancements across various engineering fields have grown demand for highly efficient materials for specific applications. Laminated composite materials have emerged as a promising solution, offering a combination of favorable mechanical properties by layering different materials. Theories, such as zig-zag theories, have been developed to predict the mechanical behavior of these materials under external stresses. This study investigates the outcomes provided by the Rayleigh-Ritz variational method, employing different approximation functions, particularly focusing on unified higher-order zig-zag kinematics. All proposed shape functions agree to the reference values, but the model with polynomial shape function shows more applications possibilities.

Keywords: composite materials, composite laminated beams, zig-zag theory, Rayleigh-Ritz method.

1 Introduction

Composite materials combine two or more materials to create a new, beneficial material. This fusion can occur at either a macroscopic or microscopic scale, aiming to enhance various properties, including strength, stiffness, corrosion resistance, aesthetics, weight, fatigue life, and thermal and acoustic behavior [2]. Consequently, research on composites has expanded, particularly in investigating their structural performance when used in applications like laminated composite beams.

To describe the mechanical behavior of a beam structure, one has theories that can be divided into classical and non-classical (or higher-order). The classical theory (or Euler-Bernoulli theory) proposes a relation between the axial displacement and the transversal displacement without the influence of the shear effect, which means that this theory cannot calculate the shear stress field [11]. This problem can be mitigated using Timoshenko theory [15] because shear influence is considered through a linear function along the beam's thickness, resulting in a constant shear stress field, which is a good improvement but does not correspond to reality. The evolution of this analysis results in the development of several high-order theories, which approximate the shear behavior by using high-order polynomials [4, 7], trigonometric [16], hyperbolic [13, 1], or exponential [3] functions, improving the results substantially.

For the mechanical analysis of laminated composite beams, the most commonly employed theories are equivalent single layer, zig-zag, and layerwise. Equivalent single layer theory, although widely used, has limitations in capturing the characteristics of individual, as it simplifies the material behavior through a basic averaging approach. Layerwise theory overcomes this limitation by providing a more detailed representation of the beam behavior, but it also increases the number of unknowns, leading to higher computational costs. As an alternative to these theories, the zig-zag theory enriches the kinematics of the equivalent single layer without the computational complexity associated with layerwise, while maintaining good precision [12].

Numerous studies have explored the application of the zig-zag laminated beam theory. Murakami *et al.* [5] introduced the concept of zig-zag as a geometric descriptor, and incorporated it into a first-order kinematic framework [15]. Tessler *et al.* [14] furthered this work by developing the refined zig-zag theory, which considers geometric aspects and material properties to determine stress fields. Vidal and Polit [17] demonstrated improved results by combining the zig-zag function of [5] with sinusoidal higher-order beam kinematics. Subsequently, researchers introduced higher-order zig-zag theories with advanced beam kinematics, enhancing accuracy, as illustrated in [6].

This study aims to analyze the efficiency of using the Rayleigh-Ritz variational method [8], employing various shape functions by using a formulation that combines zig-zag functions with a unified kinematic approach for beams, which can be of high-order or otherwise. This analysis becomes important to observe in which mechanical situations the method is most accurate, which shape function performs best for this type of problem and the possibility of using the method for more boundary conditions. The structure of this work is as follows: In Section 2, the necessary domain and boundary restrictions are established to ensure that the first variation of the total energy functional is equal to zero. In Section 3, the characteristics of the solved problem, the shape functions are presented and the convergence results are analyzed. Finally, in Section 4, the conclusions drawn from this research are presented clearly.

2 Mathematical Development

2.1 Geometrical description

Consider a laminated composite beam with length $L = x_b - x_a$, subjected to load $q(x)$ and external forces T in the x- and z-axes, as illustrated in Figure 1.

The vertical thickness (height) of the beam is denoted as $2h$, and the thickness of the k-th layer is represented by $2h^{(k)}$, $k = 1, 2, ..., N$ indicating the layer number, also the depth is denoted as b (see Figure 2). The global coordinates of the beam are defined as $z_{(i)}$, $i = 0, 1, 2, ..., N$, such that $z_0 = -h$, $z_N = h$, and $z_{(k)} = z_{(k-1)} + 2h^{(k)}$.

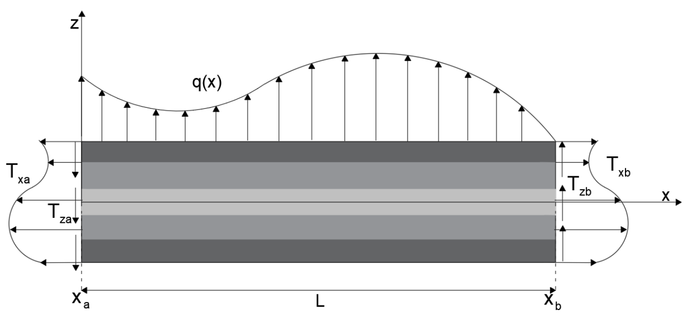

Figure 1: General loading and geometry of a composite laminated beam.

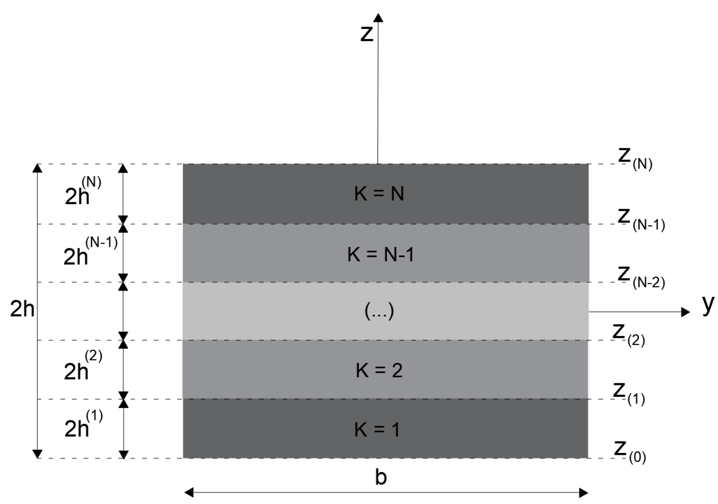

Figure 2: General cross-section of a composite laminated beam .

2.2 Kinematics

This current study is limited to the linear elastic behavior of the material. The displacement fields of various beam theories that consider shear deformation are unified through the generic function $f(z)$, which is incorporated into the kinematics defined by the k-th layer axial displacement field $u^{(k)}(x,z)$ and transverse displacement field $w(x,z)$, respectively given by

$$\begin{aligned} u^{(k)}(x,z) &= u_0(x) - z w_0'(x) + f(z)\phi(x) + \phi_{zz}^{(k)}(z)\psi(x), \\ w(x,z) &= w_0(x). \end{aligned} \tag{2.1}$$

In Eq. (2.1), $u_0(x)$ and $w_0(x)$ are the axial and transverse displacements in the midplane of the beam, respectively. Function $f(z)$ serves as a vertical shape function, contributing to both stress and shear strain higher-order distribution along the z-axis direction (see Table 1); $\phi(x)$ is the shear angle; $\phi_{zz}^{(k)}(z)$ is a zig-zag function and; $\psi(x)$ is a zig-zag amplitude function.

Table 1: Vertical shape functions for unified high-order beam theory

Source	$f(z)$
Ref. [4]	$\frac{5z}{4}\left[1-\frac{4}{3}\left(\frac{z}{2h}\right)^2\right]$
Ref. [7]	$z\left[1-\frac{4}{3}\left(\frac{z}{2h}\right)^2\right]$
Ref. [16]	$\frac{5h}{\pi}\sin(\frac{\pi z}{2h})$
Ref. [13]	$z\cosh(\frac{z}{2h}) - 2h\sinh(\frac{z}{2h})$
Ref. [3]	$z\exp\left[-2\left(\frac{z}{2h}\right)^2\right]$
Ref. [1]	$\frac{3\pi}{2}\left[2h\tanh(\frac{z}{2h}) - z\sec^2(\frac{z}{2h})\right]$

Considering a linear elastic strain field, it is possible to write the axial and shear strain fields $\varepsilon^{(k)}(x,z)$ and $\gamma^{(k)}(x,z)$ from the displacement fields in Eq. (2.1) as

$$
\begin{aligned}
\varepsilon^{(k)}(x,z) &= \frac{\partial u^{(k)}}{\partial x} = u_0'(x) - z w_0''(x) + f(z)\phi'(x) + \phi_{zz}^{(k)}(z)\psi'(x),\\
\gamma^{(k)}(x,z) &= \frac{\partial w}{\partial x} + \frac{\partial u^{(k)}}{\partial z} = f'(z)\phi(x) + \beta^{(k)}(z)\psi(x),
\end{aligned}
\tag{2.2}
$$

where $\beta^{(k)}(z)$ is the first derivative of $\phi_{zz}^{(k)}(z)$. Utilizing the linear constitutive relation for orthotropic materials [18] with elastic properties $\overline{Q_{11}}^{(k)}$ and $\overline{Q_{55}}^{(k)}$ and strain fields given by Eq. (2.2), it is possible write normal and shear stress fields $\sigma^{(k)}(x,z)$ and $\tau^{(k)}(x,z)$ as follows:

$$
\begin{aligned}
\sigma^{(k)}(x,z) &= \overline{Q_{11}}^{(k)}\varepsilon^{(k)}(x,z) = \overline{Q_{11}}^{(k)}\left[u_0'(x) - z w_0''(x) + f(z)\phi'(x) + \phi_{zz}^{(k)}(z)\psi'(x)\right],\\
\tau^{(k)}(x,z) &= \overline{Q_{55}}^{(k)}\gamma^{(k)}(x,z) = \overline{Q_{55}}^{(k)}\left[f'(z)\phi(x) + \beta^{(k)}(z)\psi(x)\right].
\end{aligned}
\tag{2.3}
$$

Within this unified kinematic framework, the term $\phi_{zz}^{(k)}(z)$ can be substituted with any zig-zag function in the literature. This can be the linear $\phi_{MUR}^{(k)}(z)$ by Murakami [5], given by

$$
\phi_{MUR}^{(k)}(z) = \frac{(-1)^k(z_{(k)} + z_{(k-1)} - 2z)}{2h^{(k)}},
\tag{2.4}
$$

or a higher-order function, such as the sinusoidal function $\phi_{SIN}^{(k)}(z)$ from [6], given by

$$
\phi_{SIN}^{(k)}(z) = \sin[\phi_{MUR}^{(k)}(z)] - \left(\frac{z^2}{2z_0} + \frac{2z - 3z_0z^2}{12z_N^2}\right)\frac{d\phi_{MUR}^{(0)}(z)}{dz} - \frac{2z^3 - 3z_0z^2}{12z_N^2}\frac{d\phi_{MUR}^{(N)}(z)}{dz}
\tag{2.5}
$$

which is employed for comparison in this study.

2.3 Rayleigh-Ritz method

This work introduces the application of the Rayleigh-Ritz method [8] to solve the differential equations that model the mechanical behavior of laminated composite beams. The formulation employed here is based on the principle of the first energy theorem [9], which states that for a conservative system, the equilibrium situation corresponds to a stationary value of the total potential energy equal to zero, commonly referred to as the weak form. The internal energy per unit volume, known as strain energy density U, characterizing the deformation of the laminated composite beam, is

$$
U(\varepsilon) = \sigma(x,z)\varepsilon(x,z),
\tag{2.6}
$$

the corresponding total differential, denoted as dU, in each layer is given by

$$dU = \frac{\partial U}{\partial \varepsilon_x} d\varepsilon_x + \frac{\partial U}{\partial \gamma_{xy}} d\gamma_{xy} = \sigma^{(k)}(x,z) d\varepsilon_x + \tau^{(k)}(x,z) d\gamma_{xy}. \tag{2.7}$$

Substitute constitutive relations in the first equalities of Eq. (2.3) into Eq. (2.7) and integrate with respect to strain to obtain the strain energy of the k-th layer as

$$U = \frac{\overline{Q_{11}^{(k)}}}{2} \left[\varepsilon^{(k)}(x,z)\right]^2 + \frac{\overline{Q_{55}^{(k)}}}{2} \left[\gamma^{(k)}(x,z)\right]^2, \tag{2.8}$$

which in turn is integrated over the domain volume after substituting the strain fields in the second equalities of Eq. (2.2) to obtain the total strain energy U_T as

$$\begin{aligned} U_T = {} & A_1 \int_0^L [\phi'(x)]^2 dx + A_2 \int_0^L \phi'(x)\psi'(x) dx + A_3 \int_0^L [\psi'(x)]^2 dx \\ & + A_4 \int_0^L \phi'(x) u_0'(x) dx + A_5 \int_0^L [u_0'(x)]^2 dx + A_6 \int_0^L \psi'(x) u_0'(x) dx \\ & - A_7 \int_0^L \phi'(x) w_0''(x) dx - A_8 \int_0^L \psi'(x) w_0''(x) dx - A_9 \int_0^L u_0'(x) w_0''(x) dx \\ & + A_{10} \int_0^L [w_0''(x)]^2 dx + A_{11} \int_0^L [\phi(x)]^2 dx + A_{12} \int_0^L \phi(x)\psi(x) dx + A_{13} \int_0^L [\psi(x)]^2 dx, \end{aligned} \tag{2.9}$$

The integrals related to the area of Eq. (2.9) can be expressed as a sum of the stiffness terms for each layer. In the context of this specific laminated beam case, these integrals are constants. These constants are denoted as A_1 to A_{10}, for stiffness related to axial deformation, A_{11} to A_{13}, for stiffness related to shear deformation, and are written as $A_n = b \sum_{k=1}^{N} \int_{z_{(k-1)}}^{z_{(k)}} S_{n(k)}(z) dz$, where the terms $S_{n(k)}(z)$ are given in Table 2.

The work Ω done by external forces F on the beam is expressed as

$$\Omega(F) = F \cdot d, \tag{2.10}$$

where, d is the displacement caused by F. In the specific case where the external load depends solely on the x-direction, it is further simplified as

$$\Omega = \int_S p(x,y) d(x,y) dS + \int_V b(x,y,z) d(x,y,z) dV = \int_L q(x) w_0(x) dx. \tag{2.11}$$

Table 2: Explicit $S_{n(k)}(z)$ terms to describe A_n stiffness

n	$S_{n(k)}$
1	$\frac{Q_{11}^{(k)}}{2}[f(z)]^2$
2	$Q_{11}^{(k)} f(z)\phi_{zz}^{(k)}(z)$
3	$\frac{Q_{11}^{(k)}}{2}[\phi_{zz}^{(k)}(z)]^2$
4	$Q_{11}^{(k)} f(z)$
5	$\frac{Q_{11}^{(k)}}{2}$
6	$Q_{11}^{(k)}\phi_{zz}^{(k)}(z)$
7	$Q_{11}^{(k)} z f(z)$
8	$Q_{11}^{(k)} z\phi_{zz}^{(k)}(z)$
9	$Q_{11}^{(k)} z$
10	$\frac{Q_{11}^{(k)}}{2} z^2$
11	$\frac{Q_{55}^{(k)}}{2}[f'(z)]^2$
12	$Q_{55}^{(k)} f'(z)\beta^{(k)}(z)$
13	$\frac{Q_{55}^{(k)}}{2}[\beta^{(k)}(z)]^2$

The energy functional $\Pi = U_T - \Omega$ depends on four variables, as

$$\Pi\left[u_0(x), w_0(x), \phi(x), \psi(x)\right] = U_T - \Omega, \tag{2.12}$$

and can also be represented by an approximating field

$$\Pi\left[u_{0(j)}, w_{0(j)}, \phi_{(j)}, \psi_{(j)}\right] \;\; , with \quad j = 1, 2, ..., m \tag{2.13}$$

weighted by appropriate longitudinal shape functions, is used for the problem variables. In Eq. (2.13), m is the number of terms in the approximating fields after truncation of the corresponding series representation. Applying the first energy theorem and setting the differential of Π to zero establishes the equilibrium conditions for a conservative system, as follow

$$d\Pi\left[u_{0(j)}, w_{0(j)}, \phi_{(j)}, \psi_{(j)}\right] = 0, \tag{2.14}$$

$$\frac{\partial \Pi}{\partial u_{0(j)}} du_{0(j)} + \frac{\partial \Pi}{\partial w_{0(j)}} dw_{0(j)} + \frac{\partial \Pi}{\partial \phi_{(j)}} d\phi_{(j)} + \frac{\partial \Pi}{\partial \psi_{(j)}} d\psi_{(j)} = 0 \ , with \quad j = 1, 2, ..., m. \tag{2.15}$$

All $4m$ unknowns in Eq. (2.14) are determined by enforcing the linearly independent condition in Eq. (2.15), as

$$\begin{aligned} du_{0(j)} \neq 0, &\quad \frac{\partial \Pi}{\partial u_{0(j)}} = 0, \\ dw_{0(j)} \neq 0, &\quad \frac{\partial \Pi}{\partial w_{0(j)}} = 0, \\ d\phi_{(j)} \neq 0, &\quad \frac{\partial \Pi}{\partial \phi_{(j)}} = 0, \\ d\psi_{(j)} \neq 0, &\quad \frac{\partial \Pi}{\partial \psi_{(j)}} = 0. \end{aligned} \tag{2.16}$$

3 Results and Discussion

3.1 Problem properties

The dimensionless expressions for the response fields of interest, indicated with subscript a, are

$$\begin{aligned} {u_a}^{(k)}(x,z) &= u^{(k)}(x,z)\frac{bE_y}{2hq_0}, \qquad w_a(x,z) = w(x,z)\frac{800bh^3E_y}{L^4q_0}, \\ {\sigma_a}^{(k)}(x,z) &= \sigma^{(k)}(x,z)\frac{b}{q_0}, \qquad {\tau_a}^{(k)}(x,z) = \tau^{(k)}(x,z)\frac{b}{q_0}. \end{aligned} \tag{3.1}$$

Other important value is the Slenderness of the beam, defined by $S = \frac{L}{2h}$. Notably, the shear stress field was computed using equilibrium equations following the procedure in [10]. The following elastic properties were used for the fiber-reinforced laminated beam:

$$E_x = 25MPa, \quad E_y = 1MPa, \quad G_{xy} = 0.5MPa, \quad G_{yz} = 0.2MPa, \quad v_{xy} = v_{yz} = 0.25. \tag{3.2}$$

The results were obtained for the problem depicted in Figure 3. Reference values were calculated in [6] using the Navier analytical solution, which combined Reddy beam kinematics [7] (see Table 1) with the sinusoidal zig-zag function in Eq. (2.5). The main objective is to analyze the sensitivity of Rayleigh-Ritz method to different types of longitudinal shape functions (as will be discussed in section 3.2) while maintaining the same combination of laminated beam theory as in [6].

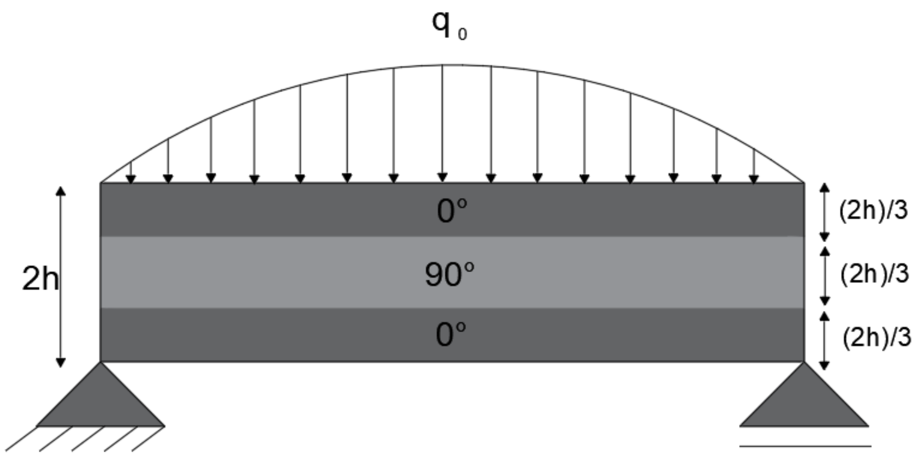

Figure 3: Proposed problem of a laminated beam simply supported.

The problem at hand concerns a simply supported laminated beam of three layers, each layer of thickness $2h/3$ and stacking sequence of $0^o/90^o/0^o$ (referring to the angle between the fibers and the x-axis in planes parallel to xy-plane). The boundary conditions for the presented problem are defined as $w(0) = w''(0) = w(L) = w''(L) = 0$. In addition, this structure has a slenderness ratio $S = 4$ and is subjected to a sinusoidal load $q_0(x)$ as

$$q(x) = q_0 \sin(\frac{\pi x}{L}). \tag{3.3}$$

3.2 Longitudinal shape functions

Three types of longitudinal shape functions were examined to approximate the response fields. The first type is an approximation using exponential functions combined with the so-called bubble function $x(L - x)$, which is referred to as the exponential shape function and is given by

$$
\begin{aligned}
w_0(x) &= \sum_{j=1}^{m} w_{0j} x(L-x)e^{\frac{x}{jL}}, \qquad u_0(x) = \sum_{j=1}^{m} u_{0j} \frac{d}{dx}[x(L-x)e^{\frac{x}{jL}}],\\
\phi(x) &= \sum_{j=1}^{m} \phi_j \frac{d}{dx}[x(L-x)e^{\frac{x}{jL}}], \qquad \psi(x) = \sum_{j=1}^{m} \psi_j \frac{d}{dx}[x(L-x)e^{\frac{x}{jL}}].
\end{aligned} \tag{3.4}
$$

In the second type, the bubble function is enhanced as $x^j(L-x)$ instead of using exponential functions, resulting in the polynomial shape function, which is given by

$$
\begin{aligned}
w_0(x) &= \sum_{j=1}^{m} w_{0j} x^j(L-x), \qquad u_0(x) = \sum_{j=1}^{m} u_{0j} \frac{d}{dx}\left[x^j(L-x)\right],\\
\phi(x) &= \sum_{j=1}^{m} \phi_j \frac{d}{dx}\left[x^j(L-x)\right], \qquad \psi(x) = \sum_{j=1}^{m} \psi_j \frac{d}{dx}\left[x^j(L-x)\right].
\end{aligned} \tag{3.5}
$$

These bubble functions are compatible with the specified boundary conditions and can be readily adapted for other problem domains. The third type of longitudinal shape function is the trigonometric shape function given by

$$
\begin{aligned}
w_0(x) &= \sum_{j=1}^{m} w_{0j} sin\left(\frac{j\pi x}{L}\right), \qquad u_0(x) = \sum_{j=1}^{m} u_{0j} \frac{d}{dx}\left[sin\left(\frac{j\pi x}{L}\right)\right],\\
\phi(x) &= \sum_{j=1}^{m} \phi_j \frac{d}{dx}\left[sin\left(\frac{j\pi x}{L}\right)\right], \qquad \psi(x) = \sum_{j=1}^{m} \psi_j \frac{d}{dx}\left[sin\left(\frac{j\pi x}{L}\right)\right].
\end{aligned} \tag{3.6}
$$

Unlike the two previous ones, this type of longitudinal shape function does not rely on bubble functions but successfully satisfies the boundary conditions.

3.3 Main results

The results for transverse displacement $w(x,0)$, axial displacement $u(L,z)$, normal stress $\sigma(L/2,z)$, and shear stress $\tau(0,z)$ were calculated for different numbers of approximation terms m until convergence was achieved to the third decimal place. This allowed for the evaluation of the accuracy of each longitudinal shape function, as depicted in Figures 4-6.

Figure 4 shows that the axial and transverse displacement fields $w(x,0)$ and $u(L,z)$ exhibit rapid convergence with the exponential shape function approximation ($m=8$), whereas the normal stress field $\sigma(L/2,z)$ necessitates four additional terms ($m=12$) to achieve convergence. However, the exponential shape function approximation exhibits slower convergence for the shear stress field $\tau(0,z)$, which poses a limitation in the context of this particular problem.

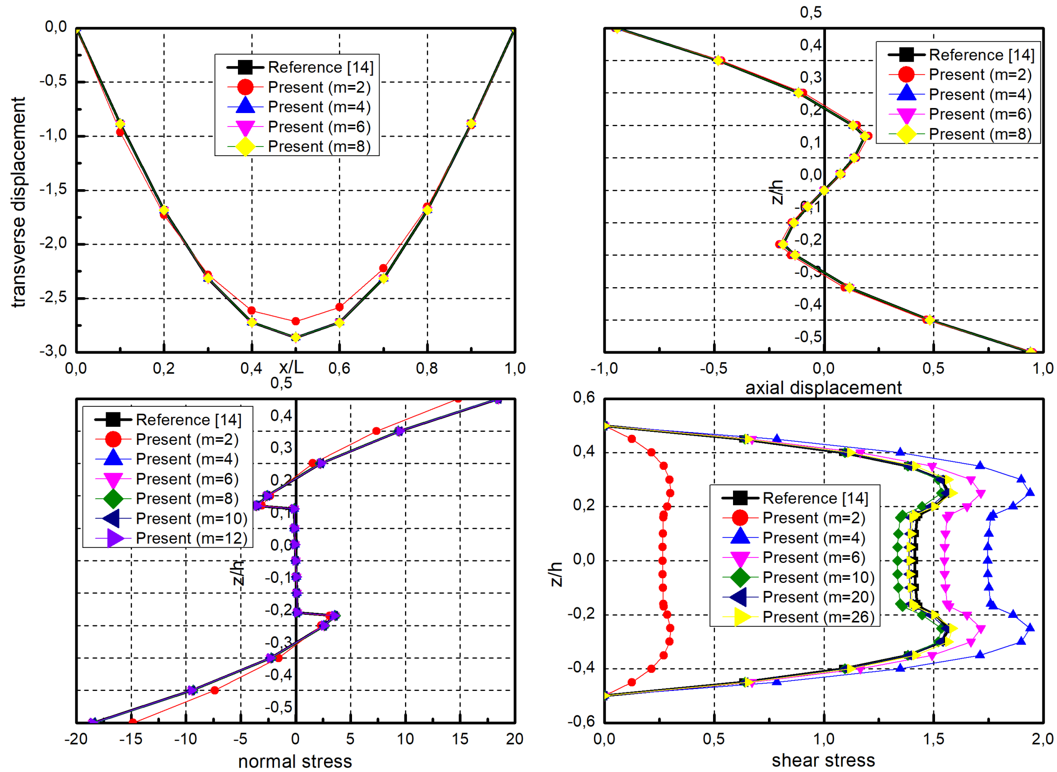

Figure 4: Results for the exponential shape function: (a) transverse displacement $w_a(x, 0)$, (b) axial displacement $u_a(L, z)$, (c) normal stress $\sigma_a(L/2, z)$ and (d) shear stress $\tau_a(0, z)$.

Figure 5 illustrates the performance of the polynomial shape function approximation. Notably, uniform convergence is observed for all response fields ($m = 8$), indicating higher accuracy in this approximation method for laminated beam problems, particularly concerning the shear stress field.

Figure 6 shows that achieving convergence required only a single term of the trigonometric shape function approximation. This is attributed to the similarity in behavior between the adopted shape function and the reference function from [6].

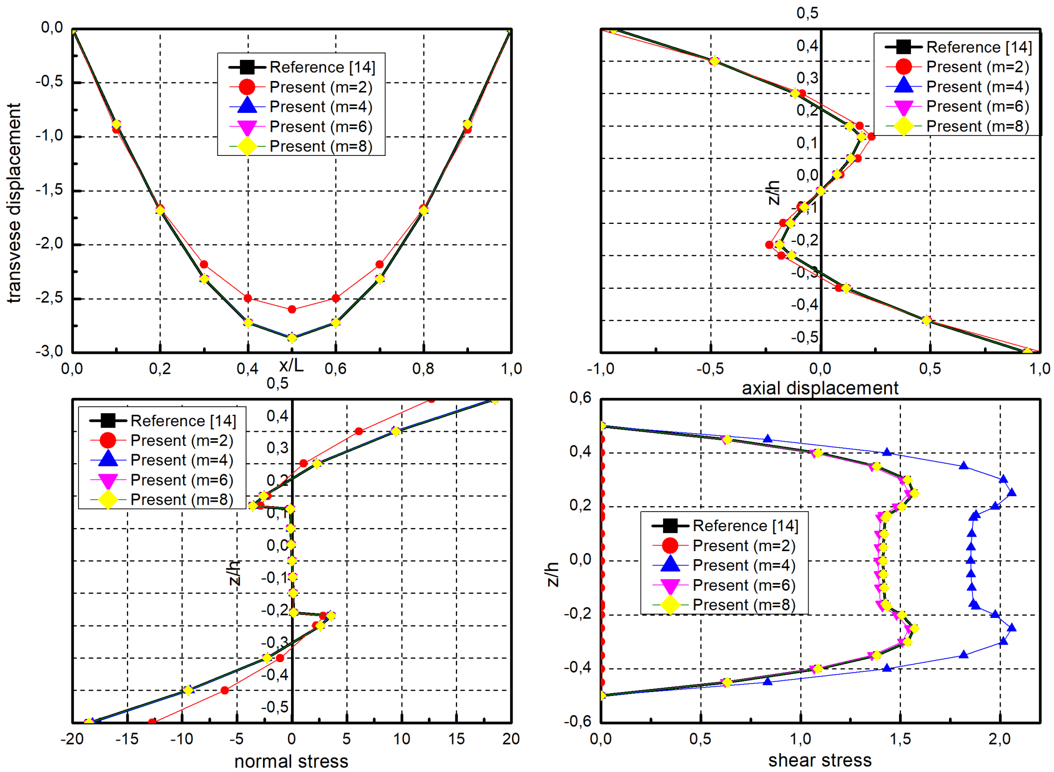

Figure 5: Results for the polynomial shape function: (a) transverse displacement $w_a(x, 0)$, (b) axial displacement $u_a(L, z)$, (c) normal stress $\sigma_a(L/2, z)$ and (d) shear stress $\tau_a(0, z)$.

4 Conclusions

Based on the results, it is possible to notice that the formulation using Rayleigh-Ritz method is precise for applications in composite laminated beam problems and offers several possibilities of combining kinematic and zig-zag functions. These results can be achieved with any longitudinal shape functions indicated for beam problems. From the three types of longitudinal shape function studied, the exponential shape function has convergence with different efficiency for each response field, presenting good results even with low m values for $w(x, z)$ and $u(x, z)$, but it shows difficulties for the stress fields, mainly shear stress. On the other hand, the polynomial shape function has uniform convergence for all response fields with $m = 8$. Finally, the trigonometric shape function presents the best convergence result for this example“, but it shows limitations to be expanded to other boundary conditions, whereas the first two types of

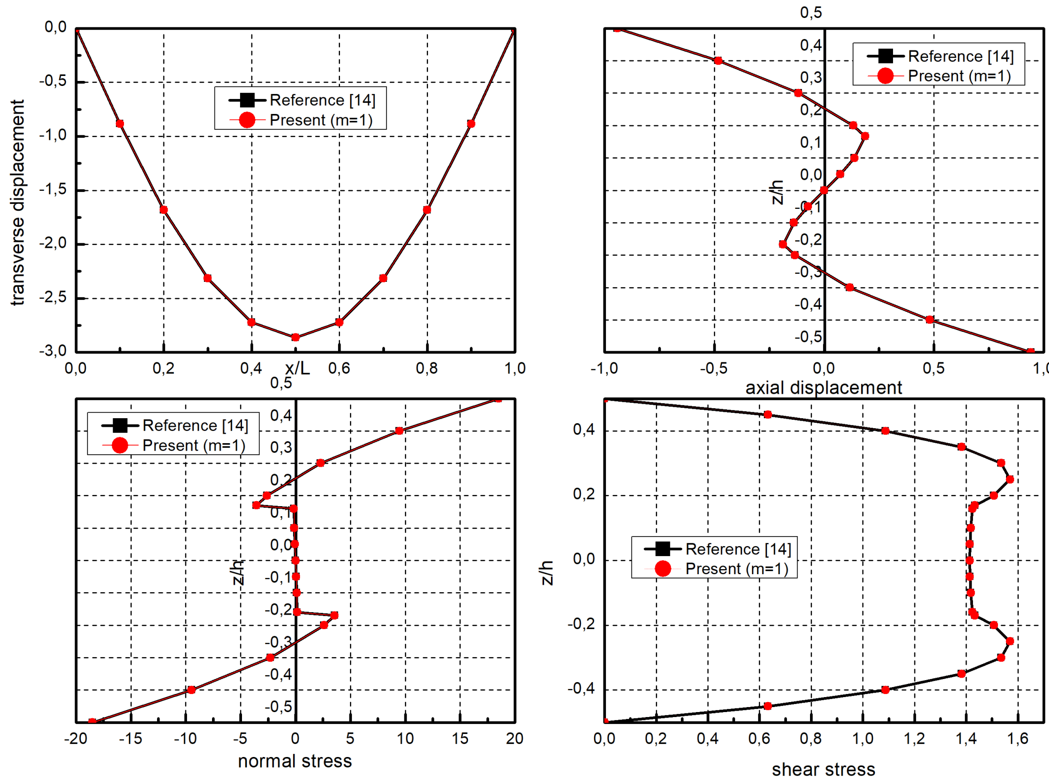

Figure 6: Results for the trigonometric shape function: (a) transverse displacement $w_a(x,0)$, (b) axial displacement $u_a(L,z)$, (c) normal stress $\sigma_a(L/2,z)$ and (d) shear stress $\tau_a(0,z)$.

approximation have the advantage of adaptability to various boundary conditions due to their composition, which include the so-called bubble functions.

Acknowledgments

The authors thank CNPq (Process No 402857/2021 – 6) and CAPES for the financial support and the Postgraduate Program in Civil Engineering at the Federal University of Sergipe (PROEC/UFS) for providing the necessary infrastructure.

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Received in 22 October 2023.
1ª revision in 14 November 2023.
Accept in 11 December 2023.