Revista Sergipana de Matemática e Educação Matemática

https://doi.org/10.34179/revisem.v6i1.19912

ON GENERALIZED ORESME NUMBERS AND THE FIBONACCI FUNDAMENTAL SYSTEM

Elen Viviani Pereira Spreafico
Universidade Federal de Mato Grosso do Sul
elen.spreafico@ufms.br
Paula Maria Machado Catarino
Universidade de Trás-os-Montes e Alto Douro
p.catarin@utad.pt

Abstract

In this work we study the generalized Oresme numbers using the algebraic properties of the Fibonacci fundamental system associated with this sequence. We explicitly describe the closed connection between the sequences of the Fibonacci fundamental system and the generalized Oresme numbers. In addition, the matrix approach is considered for studying the combinatorial identities and the generalized Cassini identity for the generalized Oresme numbers. Finally, the analytic representation is established, and the combinatorial and analytical generalized Cassini identity is given.

Keywords: Generalized sequence of numbers; Fibonacci fundamental system; combinatorial identities; the Cassini identity.

1 Introduction

A numerical sequence is a set of numbers that follows a specific pattern or rule. Whether you are a math enthusiast or just curious about numbers, sequences offer a captivating journey of patterns, trends, and infinite possibilities. From the famous Fibonacci sequence to Pell, Lucas, Jacobsthal, Padovan, and Perrin numbers, along with their generalizations, numerous sets of number sequences have been researched. One of these interesting sequences is that it consists of the Oresme numbers denoted by $\{O_n\}_{n\geq 0}$ and defined by the recurrence relation

$$O_{n+1} = O_n - \frac{1}{4}O_{n-1},\ (n \geq 1), \tag{1.1}$$

with initial conditions $O_0 = 0$ and $O_1 = \dfrac{1}{2}$.

This sequence of rational numbers was introduced by Nicole Oresme in the mid-fourteenth century. Such a sequence is of considerable biological interest as an answer to the question involving the calculus of the proportions in any later generations if the proportion of grandparents and parents of different genotypes is known.

The Oresme sequence and their generalizations are studied in several aspects, as analytic and matrix perspectives (see more in [6–10]). In the present study we discuss properties of generalized Oresme numbers given in [9], defined by a general linear recursive relation,

$$\mathbb{O}_n = p\mathbb{O}_{n-1} - q\mathbb{O}_{n-2},\ (n \geq 2), \tag{1.2}$$

where p, q are arbitrary rational numbers and initial conditions $\mathbb{O}_0 = \alpha_0$ and $\mathbb{O}_1 = \alpha_1$. Note that, if we take $p = 1$ and $q = \frac{1}{4}$, for initial conditions $\alpha_0 = 0$ and $\alpha_1 = \dfrac{1}{2}$, we have the Oresme numbers, i.e., $\mathbb{O}_n = O_n$. Then the recursive sequence $\{\mathbb{O}_n\}_{n\geq 0}$ is a generalization of Oresme sequence of numbers. Table bellow describes some sequences derived from Equation (1.2).

Table 1: Sequences derived from Equation (1.2)

p	q	α_1	α_2	Sequence
1	$\frac{1}{4}$	0	$\frac{1}{2}$	Oresme
1	-1	0	1	Fibonacci
1	-1	2	1	Lucas
2	-1	0	1	Pell
1	-2	0	1	Jacobsthal

Our objective is to investigate the generalized Oresme numbers defined by the recurrence relation (1.2), through the properties of the Fibonacci fundamental systems related to the elements of this sequence.

The Fibonacci fundamental system is the set of sequences defined by linear recursive relations of the Fibonacci type and it is a powerful tool to provide new properties and identities. Its general form was introduced and studied by Rachidi *et al.* in [1, 5]. In addition, it was used in other studies, such as [11] for deriving results for generalized Fibonacci numbers, in [12] for exploring generalized Pell numbers, in [3] for investigating generalized Pell model and in [4] establishing the generalized Cassini identities.

This paper is organized as follows: Section 1 concerns the generalized Oresme numbers and its associated Fibonacci fundamental system. We establish some properties of the Fibonacci fundamental system related to generalized Oresme numbers, we provide the combinatorial aspects of generalized Oresme numbers and we study the analytical formulas for the generalized Oresme numbers. Section 2 is devoted to studying the

Casoratian matrix and providing identities for the generalized Oresme numbers. Section 3 is concerned with discussing the Cassini identity for Oresme numbers, and also the combinatorial and the analytical formula of the generalized Cassini identity are established. Finally, some conclusions and perspectives are stated.

2 The generalized Oresme numbers and Fibonacci Fundamental System

The Fibonacci fundamental system is a fascinating tool for analyzing sequences that are defined by linear recursive relations of the Fibonacci type. This approach has been successfully applied to model generalized sequences, such as generalized Fibonacci numbers and generalized Pell numbers (for more information, see [3, 11, 12]). Through extensive research, numerous results have been established in this field. In the next subsections, we present the Fibonacci fundamental system related to the generalized Oresme numbers and establish some properties.

2.1 Fibonacci fundamental system related to the generalized Oresme numbers

Consider the sequences $\{\mathbb{O}_n^{(1)}\}_{n\geq 0}$, and $\{\mathbb{O}_n^{(2)}\}_{n\geq 0}$ defined by

$$\begin{aligned} \mathbb{O}_{n+1}^{(j)} &= p\mathbb{O}_n^{(j)} - q\mathbb{O}_{n-1}^{(j)}, \text{ for } n \geq 1, \\ \mathbb{O}_n^{(j)} &= \delta_{n+1}^{(j)}, \text{ for } n = 0,\, 1, \end{aligned} \tag{2.1}$$

with p, q any arbitrary rational numbers and $\delta_i^{(j)} = 1$ if $i = j$ and $\delta_i^{(j)} = 0$, otherwise. As an interesting example if we consider $p = 1$ and $q = \frac{1}{4}$, we obtain the sequences $\{\mathbb{O}_n^{(1)}\}_{n\geq 0}$, and $\{\mathbb{O}_n^{(2)}\}_{n\geq 0}$ defined by $\mathbb{O}_{n+1}^{(1)} = \mathbb{O}_n^{(1)} - \frac{1}{4}\mathbb{O}_{n-1}^{(1)}$, for $n \geq 1$, with $\mathbb{O}_0^{(1)} = 1$, $\mathbb{O}_1^{(1)} = 0$ and $\mathbb{O}_{n+1}^{(2)} = \mathbb{O}_n^{(2)} - \frac{1}{4}\mathbb{O}_{n-1}^{(2)}$, for $n \geq 1$, with $\mathbb{O}_0^{(2)} = 0$ and $\mathbb{O}_1^{(2)} = 1$. This example consists of the Fibonacci fundamental system related to the Oresme numbers given by recurrence relation (1.1).

Let $\mathscr{E}_{\mathbb{O}}$ be the set of sequences $\{\mathbb{O}_n\}_{n\geq 0}$ satisfying the recurrence relation (1.2) and the set $\mathscr{S}_{\mathbb{O}} = \left\{\{\mathbb{O}_n^{(1)}\}_{n\geq 0};\ \{\mathbb{O}_n^{(2)}\}_{n\geq 0}\right\}$. We can prove that the set $\mathscr{E}_{\mathbb{O}}$ of sequences satisfying the recursive relation (1.2) is a real vector space. Moreover, the next result shows us that the set $\mathscr{S}_{\mathbb{O}}$ is a generator system of $\mathscr{E}_{\mathbb{O}}$.

Proposition 2.1. *Let* $\{\mathbb{O}_n\}_{n\geq 0}$ *be the sequence defined by the recursive relation (1.2) with arbitrary initial conditions* α_0 *,*α_1*. Then, for every* $n \geq 0$*, we have,*

$$\mathbb{O}_n = \alpha_0 \mathbb{O}_n^{(1)} + \alpha_1 \mathbb{O}_n^{(2)}.$$

Proof. Let $\{w_n\}_{n\geq 0}$ be the sequence defined by, $w_n = \alpha_0 \mathbb{O}_n^{(1)} + \alpha_1 \mathbb{O}_n^{(2)}$, for every $n \geq 0$, with initial conditions (α_0, α_1). For every $n \geq 2$, we have,

$$\begin{aligned} w_n &= \sum_{j=0}^{1} \alpha_j \mathbb{O}_n^{(j+1)} = \sum_{j=0}^{1} \alpha_j [p\mathbb{O}_{n-1}^{(j+1)} - q\mathbb{O}_{n-2}^{(j+1)}] \\ &= pw_{n-1} - qw_{n-2}. \end{aligned}$$

Then, the sequence $\{w_n\}_{n\geq 0}$ satisfies the recursive relation (1.2). Moreover, we have the initial conditions,

$$\begin{aligned} w_0 = \alpha_0 \mathbb{O}_0^{(1)} + \alpha_1 \mathbb{O}_0^{(2)} = \alpha_0, \\ w_1 = \alpha_0 \mathbb{O}_1^{(1)} + \alpha_1 \mathbb{O}_1^{(2)} = \alpha_1, \end{aligned}$$

Therefore, for every $n \geq 0$, we have $\mathbb{O}_n = w_n = \alpha_0 \mathbb{O}_n^{(1)} + \alpha_1 \mathbb{O}_n^{(2)}$.

□

Proposition 2.1 gives us that $\mathscr{S}_O = \left\{ \{\mathbb{O}_n^{(1)}\}_{n\geq 0};\ \{\mathbb{O}_n^{(2)}\}_{n\geq 0} \right\}$ is the *Fibonacci fundamental system* related to generalized Oresme numbers.

2.2 Properties of the Fibonacci fundamental system related to generalized Oresme numbers

In this subsection, we give the closed connection between the sequences $\{\mathbb{O}_n^{(1)}\}_{n\geq 0}$, $\{\mathbb{O}_n^{(2)}\}_{n\geq 0}$, and provide some properties. More precisely, we have the following result.

Proposition 2.2. *Let* $\mathscr{S}_O = \left\{ \{\mathbb{O}_n^{(1)}\}_{n\geq 0};\ \{\mathbb{O}_n^{(2)}\}_{n\geq 0} \right\}$ *be the Fibonacci fundamental system related to generalized Oresme numbers (1.2). Then we have,*

$$\mathbb{O}_n^{(1)} = -q\mathbb{O}_{n-1}^{(2)} \tag{2.2}$$

for $n \geq 1$*.*

Proof. Let w_n be the sequence defined by $w_n = -q\mathbb{O}^{(2)}_{n-1}$, $n \geq 1$. Note that for $n = 1$ we have $w_1 = -q\mathbb{O}^{(2)}_0 = 0$ and for $n = 2$ we obtain $w_2 = -q\mathbb{O}^{(2)}_1 = -q$. Moreover, $pw_{n+1} - qw_n = -q(p\mathbb{O}^{(2)}_n - q\mathbb{O}^{(2)}_{n-1}) = -q\mathbb{O}^{(2)}_{n+1} = w_{n+2}$, i.e, the sequence w_n satisfies the recurrence relation (1.2). Then, $\mathbb{O}^{(1)}_n = w_n = -q\mathbb{O}^{(2)}_{n-1}$, for $n \geq 1$. □

As a consequence, Propositions 2.1 and 2.2 permit us to obtain the following general result, which the proof is omitted.

Proposition 2.3. *Let $\{\mathbb{O}_n\}_{n\geq 0}$ be the sequence defined by the recursive relation (1.2) with initial conditions α_0, α_1. Then, for every $n \geq 0$, we have,*

$$\mathbb{O}_n = -q\alpha_0\mathbb{O}^{(2)}_{n-1} + \alpha_1\mathbb{O}^{(2)}_{n-2}. \tag{2.3}$$

As an interesting example by replacing $p = 1$ and $q = \frac{1}{4}$, and initial conditions $\alpha_0 = 0, \alpha_1 = \frac{1}{2}$, in Equation (2.3), we obtain the Oresme numbers $\{\mathbb{O}_n\}_{n\geq 0}$, in terms of the fundamental solution, $\{\mathbb{O}^{(2)}_n\}_{n\geq 0}$.

Corollary 2.1. *Consider the sequence of Oresme numbers denoted by $\{\mathbb{O}_n\}_{n\geq 0}$ and given by recurrence relation (1.1) . Then, for every $n \geq 0$, we have,*

$$\mathbb{O}_n = \frac{1}{2}\mathbb{O}^{(2)}_{n-2}.$$

2.3 Combinatorial formulas for the generalized Oresme numbers

In this section, we will discuss the combinatorial expressions of the generalized Oresme numbers. The combinatorial expressions of Fibonacci type sequences $\{u_n\}_{n\geq 0}$ defined by recurrence relation $u_{n+1} = \sum_{i=0}^{r-1} a_i u_{n-i-1}$, for $n \geq r$, have been largely explored and some interesting results were provided (see more in [3, 11, 12], and references therein). First, consider the following combinatorial expression,

$$\rho(n,2) = \sum_{m_0+2m_1=n-2} \frac{(m_0+m_1)!}{m_0!m_1!} a_0^{m_0} a_1^{m_1}, \text{ for every } n \geq 2, \tag{2.4}$$

where a_0, a_1 are real or complex numbers, $\rho(0,2) = \rho(1,2) = 0$ and $\rho(2,2) = 1$. We can show that

$$\rho(n+1,2) = a_0\rho(n,2) + a_1\rho(n-1,2).$$

Specially, for $a_0 = p$ and $a_1 = -q$, for every $n \geq 1$, we get

$$\rho(n+1,2) = (-1)^{m_1} \sum_{m_0+2m_1=n-2} \frac{(m_0+m_1)!}{m_0!m_1!} p^{m_0} q^{m_1}, \tag{2.5}$$

with $\rho(0,2) = \rho(1,2) = 0$ and $\rho(2,2) = 1$. Therefore, the sequence $\{\rho(n,2)\}_{n\geq 0}$ satisfies the Expression (1.2) and its initial conditions are given by $\rho(0,1) = \rho(1,2) = 0$ and $\rho(2,2) = 1$. Thus we can formulate the combinatorial expression of $(\mathbb{O}_n^{(2)})_{n\geq 0}$, given in the next result.

Proposition 2.4. *The combinatorial expression of* $(\mathbb{O}_n^{(2)})_{n\geq 0}$ *is given by*

$$\mathbb{O}_n^{(2)} = \rho(n+1,2) = (-1)^{m_1} \sum_{m_0+2m_1=n-2} \frac{(m_0+m_1)!}{m_0!m_1!} p^{m_0} q^{m_1}, \tag{2.6}$$

for every $n \geq 2$, *where* $\mathbb{O}_0^{(2)} = \rho(1,2) = 0$ *and* $\mathbb{O}_1^{(2)} = \rho(2,2) = 1$.

Combining the Propositions 2.3 and 2.4 we obtain the following general result.

Proposition 2.5. *Let* $\{\mathbb{O}_n\}_{n\geq 0}$ *be the sequence defined by the recursive relation (1.2) and with initial conditions* α_0, α_1. *Then, for every* $n \geq 0$, *we have,*

$$\mathbb{O}_n = -q\alpha_0\rho(n,2) + \alpha_1\rho(n-1,2). \tag{2.7}$$

Similarly, by replacing $p = 1$ and $q = \frac{1}{4}$, and initial conditions $\alpha_0 = 0, \alpha_1 = \frac{1}{2}$, in Equation (2.7), we obtain a combinatorial expression for Oresme numbers $\{\mathbb{O}_n\}_{n\geq 0}$, stated in the following corollary.

Corollary 2.2. *Let* $\{\mathbb{O}_n\}_{n\geq 0}$ *be the sequence of Oresme numbers. Then, for every* $n \geq 0$, *we have,*

$$\mathbb{O}_n = \frac{1}{2}\rho(n-1,2) = \frac{1}{2}(-1)^{m_1} \sum_{m_0+2m_1=n-2} \frac{(m_0+m_1)!}{m_0!m_1!} p^{m_0} q^{m_1}.$$

It seems to us that these combinatorial formulas are not current in the literature.

2.4 Analytical aspect of the generalized Oresme numbers

In this section, we will use the Sylvester matrix approach (see more in [3]) to study the roots of the characteristic polynomials of the generalized recursive equation (1.2) given by $P(z) = z^2 - pz + q$. Sylvester matrix is defined as a matrix associated with two univariate polynomials $P(z)$ and $Q(z)$, whose entries are given by coefficients of these two polynomials with the following property: when the determinant of the Sylvester matrix $S_{P,Q}$, called the resultant, is zero, then the two polynomials have a common root. Therefore, by considering the polynomial $P(z)$ and its derivative $P'(z)$, if the determinant of the Sylvester matrix $S_{P,P'}$ is different from 0, then the polynomials $P(z)$ and $P'(z)$ do not have common roots. This means that if $det(S_{P,P'}) \neq 0$, then the roots of $P(z)$ are simple. In this case, the associated characteristic polynomial is $P(z) = z^2 - pz + q$. with derivative $P'(z) = 2z - p$, and its Sylvester matrix associated is given by,

$$S_{P,P'} = \begin{pmatrix} 1 & -p & q \\ 2 & -p & 0 \\ 0 & 2 & -p \end{pmatrix}.$$

Hence, $det(S_{P,P'}) = -p^2 + 4q$. Then $det(S_{P,P'}) = 0$, if and only if $p^2 = 4q$. Observe that the determinant equals the discriminant of $P(z)$ to the minus of the sign. However, the Sylvester matrix approach must be considered for analysis in cases of order greater than 2. Under the preceding discussion, we get the following analytical result.

Lemma 2.1. *Consider rational numbers p and q. The roots of the polynomial*

$$P(z) = z^2 - pz + q,$$

are simple if and only if $p^2 \neq 4q$.

Since the sequence $\{\rho(n,2)\}_{n\geq 0}$ defined by (2.4) is expressed in an analytic form in terms of the roots of characteristic polynomial (see [2] and references therein), then next result can be established.

Lemma 2.2. *Let $\{\rho(n,2)\}_{n\geq 0}$ be the sequence defined by (2.4). Suppose that the roots λ_1,λ_2 of its characteristic polynomial $P(z) = z^2 - pz + q$, satisfying $\lambda_1 \neq \lambda_2$. Then, we have*

$$\rho(n,2) = \frac{\lambda_1^{n-1}}{P'(\lambda_1)} + \frac{\lambda_2^{n-1}}{P'(\lambda_2)} = \frac{\lambda_1^{n-1}}{(\lambda_1 - \lambda_2)} + \frac{\lambda_2^{n-1}}{(\lambda_1 - \lambda_2)} \text{ for every } n \geq 2,$$

with $\rho(1,2) = 0$, $\rho(2,2) = 1$, where $P'(z) = \frac{dP}{dz}(z)$.

By Lemma 2.2 and Proposition 2.4, we obtain the analytic form for $\mathbb{O}_n^{(2)}$.

Proposition 2.6. *The analytical expression of $(\mathbb{O}_n^{(2)})_{n\geq 0}$ is given by*

$$\mathbb{O}_n^{(2)} = \frac{\lambda_1^n}{(\lambda_1 - \lambda_2)} + \frac{\lambda_2^n}{(\lambda_1 - \lambda_2)} \tag{2.8}$$

for every $n \geq 2$, where λ_1,λ_2 are roots of the characteristic polynomial $P(z) = z^2 - pz + q$, satisfying $\lambda_1 \neq \lambda_2$.

By extending the Proposition 2.6 for any sequence $\{\mathbb{O}_n\}_{n\geq 0}$ defined by the recursive relation (1.2), we get the following general result,

Proposition 2.7. *Let $\{\mathbb{O}_n\}_{n\geq 0}$ be the sequence defined by the recursive relation (1.2) with initial conditions α_0, and α_1. Then, for every $n \geq 2$, we have,*

$$\mathbb{O}_n = -q\alpha_0 \sum_{i=1}^{2} \frac{\lambda_i^{n-1}}{\prod_{k\neq i}(\lambda_i - \lambda_k)} + \alpha_1 \sum_{i=1}^{2} \frac{\lambda_i^{n-2}}{\prod_{k\neq i}(\lambda_i - \lambda_k)} \tag{2.9}$$

where λ_1,λ_2 are roots of the characteristic polynomial $P(z) = z^2 - pz^2 + q$, satisfying $\lambda_1 \neq \lambda_2$.

Proof. The proof is given by direct application of the Lemma 2.2 to Proposition 2.5. □ □

Since, for the Oresme sequence we have $p^2 = 4q$, then Proposition 2.7 can not be applied. The case of roots of the characteristic polynomial with multiplicity is also studied in [1] and a similar analytical proposition can be provided.

3 Casoratian matrix and identities for the generalized Oresme numbers

In this section, we will explore the matrix formulation of Expression (1.2) and its relation with the Fibonacci fundamental system. The relation with matrices and the Fibonacci fundamental system continues to give many results (see more in [3,4,11,12]). Consider the matrix formulation of the recursive relation (1.2) given as follows,

$$W_n = AW_{n-1}, \text{ for every } n \geq 3,$$

where

$$A = \begin{pmatrix} p & -q \\ 1 & 0 \end{pmatrix} \text{ and } W_n = \begin{pmatrix} \mathbb{O}_n \\ \mathbb{O}_{n-1} \end{pmatrix}. \tag{3.1}$$

Consider the well-known Casoratian matrix, $\widehat{C}(n)$, related to the family of sequences $\{\mathbb{O}_n^{(1)}\}_{n\geq 0}$, $\{\mathbb{O}_n^{(2)}\}_{n\geq 0}$, defined by

$$\widehat{C}(n) = \begin{pmatrix} \mathbb{O}_n^{(1)} & \mathbb{O}_n^{(2)} \\ \mathbb{O}_{n+1}^{(1)} & \mathbb{O}_{n+1}^{(2)} \end{pmatrix}. \tag{3.2}$$

A direct verification shows that the Casoratian matrix can be written under the form, $\widehat{C}(n) = J \times M_n \times J$, where $J = (b_{i,j})_{1\leq i,j\leq 2}$ is the anti-diagonal unit matrix, namely, $b_{i,j} = 1$, for $i + j = 3$, and $b_{i,j} = 0$, otherwise and $M_n = (\mathbb{O}_{n+2-i-1}^{(j)})_{1\leq i,j\leq 2}$. We show that the matrix M_n, can be written under the form $M_n = A^n$, where A is the classical companion matrix given in (3.1), (for more details see, [1, 5] and references therein). Hence, we get the following property stated in the next result.

Theorem 3.1. *Let $\mathscr{S}_O = \left\{\{\mathbb{O}_n^{(j)}\}_{n\geq 0};\ 1 \leq j \leq 2\right\}$ be the Fibonacci fundamental system of the generalized Oresme numbers defined by the recurrence relation (1.2). Then, for every $n \geq 0$, the entries of the powers $A^n = (a_{ij}^{(n)})_{1\leq i,j\leq 2}$ are given under the form,*

$$a_{ij}^{(n)} = \mathbb{O}_{n+2-i}^{(2-j+1)}. \tag{3.3}$$

Since $\widehat{C}(n) = J \times M_n \times J$ implies the matrix identity $\widehat{C}(n+m) = \widehat{C}(n).\widehat{C}(m)$, for every n and m, then the entries of the matrix $\widehat{C}(n+m) = (c_{ij}^{(n+m)})_{1\leq i,j\leq 2}$, are expressed in terms of those of the matrices $\widehat{C}(m) = (c_{ij}^{(m)})_{1\leq i,j\leq 2}$ and $\widehat{C}(n) = (c_{ij}^{(n)})_{1\leq i,j\leq 2}$ as follows,

$$c_{ij}^{(n+m)} = \sum_{k=1}^{2} c_{ik}^{(n)} c_{kj}^{(m)} = \sum_{k=1}^{2} c_{ik}^{(m)} c_{kj}^{(n)}, \text{ for every } n,\ m \geq 0, \tag{3.4}$$

where $1 \leq i,\ j \leq 2$.

Expression (3.4) and Theorem 3.1 give us the next result,

Proposition 3.1. *Let $\mathscr{S}_O = \left\{\{\mathbb{O}_n^{(1)}\}_{n\geq 0};\ \{\mathbb{O}_n^{(2)}\}_{n\geq 0}\right\}$ be the Fibonacci fundamental system of the generalized Oresme numbers defined by the recurrence relation (1.2). Then, for each $m, n \geq 0$, the following identity is verified,*

$$\mathbb{O}_{n+m+2-i}^{(2-j+1)} = \sum_{k=1}^{2} \mathbb{O}_{m+2-i}^{(3-k)} \mathbb{O}_{n+2-k}^{(3-j)}. \tag{3.5}$$

In special, for $i = 2$ and $j = 1$, we obtain

$$\mathbb{O}_{n+m}^{(2)} = \sum_{k=1}^{2} \mathbb{O}_{m}^{(3-k)} \mathbb{O}_{n+2-k}^{(2)} = \mathbb{O}_{m}^{(2)} \mathbb{O}_{n+1}^{(2)} + \mathbb{O}_{m}^{(1)} \mathbb{O}_{n}^{(2)}.$$

By Expression (2.2) of Proposition 2.2, namely, $\mathbb{O}_{n}^{(1)} = -q\mathbb{O}_{n-1}^{(2)}$, we obtain the identity,

$$\mathbb{O}_{n+m}^{(2)} = \sum_{k=1}^{2} \mathbb{O}_{m}^{(3-k)} \mathbb{O}_{n+2-k}^{(2)} = \mathbb{O}_{m}^{(2)} \mathbb{O}_{n+1}^{(2)} - q\mathbb{O}_{m-1}^{(2)} \mathbb{O}_{n}^{(2)}.$$

Hence, the previous discussion and the Corollary 2.1 allow us to establish an identity involving the Oresme numbers and the elements of the sequence $\{\mathbb{O}_{n}^{(2)}\}_{n\geq 0}$.

Corollary 3.1. *Let $\{\mathbb{O}_n\}_{n\geq 0}$ be the sequence of Oresme numbers. Then, for every $n \geq 0$, we have,*

$$\mathbb{O}_{n+m} = \frac{1}{2}(\mathbb{O}_{m-2}^{(2)} \mathbb{O}_{n+1}^{(2)} - q\mathbb{O}_{m-3}^{(2)} \mathbb{O}_{n}^{(2)}).$$

By the result of previous sections and Corollary 3.1, we obtain the following result from a combinatorial perspective.

Proposition 3.2. *Let $\{\mathbb{O}_n\}_{n\geq 0}$ be the sequence of Oresme numbers. Then, for every $n \geq 0$, we have,*

$$\mathbb{O}_{n+m} = \frac{1}{2}(\rho(m-1,2)\rho(n+2,2) - q\rho(m-2,2)\rho(n+1,2)).$$

Moreover, we can establish the analytical identity using Proposition 2.6.

4 The Cassini identity for the generalized Oresme numbers

In [4] the authors explore the generalized Cassini identities for the weighted generalized Fibonacci sequences, through the associated generalized Fibonacci fundamental system, providing the algebraic, combinatoric, and analytic formulations of the Cassini identities. This section is devoted to providing the Cassini identity for the generalized Oresme numbers via the Fibonacci fundamental system, which is an application among the results of the article [4].

Consider the matrix formulation given in (3.1). We have $W_{n+2} = AW_{n-1} = A^n W_2$. Since, it is verify $M_n = A^n$, where A is the classical companion matrix given in (3.1), then we show that

$$\det[\widehat{C}(n)] = det[A^n] = det[A]^n = (-q)^n.$$

Also,

$$\widehat{C}(n) = \begin{pmatrix} \mathbb{O}_n^{(1)} & \mathbb{O}_n^{(2)} \\ \mathbb{O}_{n+1}^{(1)} & \mathbb{O}_{n+1}^{(2)} \end{pmatrix} = \begin{pmatrix} -q\mathbb{O}_{n-1}^{(2)} & \mathbb{O}_n^{(2)} \\ -q\mathbb{O}_n^{(2)} & \mathbb{O}_{n+1}^{(2)} \end{pmatrix}, \tag{4.1}$$

and,

$$\det[\widehat{C}(n)] = -q(\mathbb{O}_{n-1}^{(2)}\mathbb{O}_{n+1}^{(2)} - (\mathbb{O}_n^{(2)})^2).$$

Therefore, we obtain the following theorem.

Theorem 4.1. *The generalized Cassini identity, related to the Fibonacci fundamental system (2.1), is given under the following form,*

$$\mathbb{O}_{n-1}^{(2)}\mathbb{O}_{n+1}^{(2)} - (\mathbb{O}_n^{(2)})^2 = (-1)^{n+1}q^{n-1}.$$

As a consequence, Expression (2.6) and Theorem 4.1 permit us to obtain the combinatorial generalized Cassini identity related to the Fibonacci fundamental system (2.1),

Proposition 4.1. *The combinatorial generalized Cassini identity, related to the Fibonacci fundamental system (2.1), is given under the following form,*

$$\rho(n,2)\rho(n+2,2) - (\rho(n+1,2))^2 = (-1)^{n+1}q^{n-1}.$$

In a similar process, we can derive the Cassini identity for Oresme numbers, $\{\mathbb{O}_n\}_{n\geq 0}$. In fact, since

$$\begin{pmatrix} \mathbb{O}_{n+1} & \mathbb{O}_n \\ \mathbb{O}_n & \mathbb{O}_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & -\frac{1}{4} \\ 1 & 0 \end{pmatrix}^{n-1} \begin{pmatrix} \mathbb{O}_2 & \mathbb{O}_1 \\ \mathbb{O}_1 & \mathbb{O}_0 \end{pmatrix}.$$

Thus, a direct computation allows us to obtain the following result,

Corollary 4.1. *Let $\{\mathbb{O}_n\}_{n\geq 0}$ be the sequence of Oresme numbers. Then, for every $n \geq 3$, it is verified,*

$$\mathbb{O}_{n+1}\mathbb{O}_{n-1} - (\mathbb{O}_n)^2 = (-1)^n \frac{1}{4}^n.$$

In addition, we can establish the combinatorial and analytical Cassini identity by using the results of Propositions 2.4 and 2.6.

5 Conclusions

In this paper, we study the generalized Oresme numbers using the approach of the Fibonacci fundamental system. In addition, some identities and combinatorial identities for the generalized Oresme numbers are provided. Moreover, we presented a study of the characteristic polynomial associated with the generalized Oresme numbers and provided analytic formulas.

It seems that several results of our study are new in the literature.

Acknowledgments

The first author is CNPq scholarship holder Process 2007770/2022-5 and expresses their sincere thanks to the Brazilian National Council for Scientific and Technological Development- CNPq- Brazil and Federal University of Mato Grosso do Sul - UFMS/MEC - Brazil for their valuable support. The second author is member of the Research Centre CMAT-UTAD (Polo of Research Centre CMAT - Centre of Mathematics of University of Minho) and thanks the Portuguese Funds through FCT – Fundação para a Ciência e a Tecnologia, within the Projects UIDB/00013/2020 and UIDP/00013/2020.

References

[1] Ben Taher, Rajae; Rachidi, Mustapha: On the matrix powers and exponential by r-generalized Fibonacci sequences methods: the companion matrix case. Linear Algebra Appl. **370** (2003), 341-353.

[2] Ben Taher, Rajae; Rachidi, Mustapha: Solving some generalized Vandermonde systems and inverse of their associate matrices via new approaches for the Binet formula. Appl. Math. Comput. **290,** (2016), 267-280.

[3] Craveiro, Irene M.; Pereira Spreafico, Elen V.; Rachidi, Mustapha: On a model of generalized Pell numbers. International Journal of Advanced Engineering Research and Science (IJAERS) **8** (2021), no. 9, 527-550.

[4] Craveiro, Irene M.; Pereira Spreafico, Elen V.; Rachidi, Mustapha: Generalized Cassini identities via the generalized Fibonacci fundamental system. Applications. Indian J Pure Appl Math (2023). https://doi.org/10.1007/s13226-023-00430-1

[5] El Wahbi, Bouazza; Mouline, Mehdi; Rachidi, Mustapha: Solving nonhomogeneous recurrences relations by matrix method. Fibonacci Quart. **40** (2002), no. 2, 109-117.

[6] Halıcı, Serpil; Sayin, Elifcan: On some k-Oresme hybrid numbers. Utilitas Mathematica, **120** (2023), 1-11.

[7] Halıcı, Serpil and Gür, Zehra Betül. On Some Derivatives of k-Oresme Polynomials. Bull. Int. math. Virtual Inst. **13** (2023), no. 1, 41-50.

[8] Halıcı, Serpil and Gür, Zehra Betül and Sayın, Elifcan. k-Oresme polynomials and their derivatives. *Mathematical methods for engineering applications*, 201–210. Springer Proc. Math. Stat., 414 Springer, Cham, 2023.

[9] Horadam, A. F.: Oresme numbers. Fibonacci Quart. **12** (1974), 267-271.

[10] Mangueira, Milena C. S.; Vieira, Remata P. M.; Alves, Francisco R. V.; Catarino, Paula M. M.: The Oresme sequence: The generalization of its matrix form and its hybridization process. Notes on Number Theory and Discrete Mathematics, **27** (2021), no. 1, 101-111.

[11] Pereira Spreafico, Elen V.; Rachidi, Mustapha: Fibonacci Fundamental System and Generalized Cassini Identity. Fibonacci Quart. **57** (2019), no. 2, 155-167.

[12] Pereira Spreafico, E.V.; Rachidi ,M.: On Generalized Pell Numbers of Order $r \geq 2$. Trends Comput. Appl. Math. **22** (2021), no. 1, 125-138.

Received 03 November 2023.
Accepted 06 February 2024.