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RAYLEIGH-RITZ METHOD APPLIED TO HIGH-ORDER ZIGZAG THEORY FOR STATIC ANALYSIS OF LAMINATED COMPOSITE BEAMS WITH DIFFERENT BOUNDARY CONDITIONS

Abstract

Laminated composite beams are structural elements that join layers with different materials to achieve the expected mechanical behavior. The zigzag theories seek to approximate such behavior, but most analyses are generally for simple supported beams. Thus, in this work, the Rayleigh-Ritz method is used to obtain the solution of the differential equation of the problem under analysis when several boundary conditions are considered. The accuracy in getting the displacement and stress fields when polynomial shape functions are used in the Rayleigh-Ritz method for the various boundary conditions adopted in this work is analyzed. The results generated by the approximate solution with a few terms in the series ($m = 8$) proved sufficient to recover the answers from the reference values.

Keywords: composite materials, composite laminated beams, zigzag theory, Rayleigh-Ritz method.

1 Introduction

According to [2], a composite material results from combining two or more materials that do not mix or dissolve into each other. This process creates an engineering material with unique properties not found in the individual components. The primary aim of this combination is to enhance various properties, including strength, stiffness, resistance to corrosion, aesthetics, weight, fatigue life, and thermal and acoustic behavior [3].

Various beam theories exist for static analysis of an isotropic structure under external loads. The first theory, Euler-Bernoulli, neglects the shear effect in its kinematics. As a result, it cannot predict the shear stress distribution and its impacts on displacements and rotations [13]. This limitation led to the development of the Timoshenko approach [17], which considers the shear effect by introducing a linear function along the height of the beam, resulting in a constant shear stress field. While this was an improvement, it still does not fully represent the real-world behavior of beams.

Subsequently, several high-order theories emerged in the pursuit of better accuracy. These theories approximate the shear behavior using high-order polynomials, trigonometric, hyperbolic, or exponential functions [5, 11, 18, 15, 4, 1], substantially improving analysis results.

When it comes to theories of laminated beams, the most frequently cited are equivalent single-layer (ESL), zigzag (ZZ), and layerwise (LW). An equivalent single layer simplifies multiple layers into one layer but does not capture the interactions within laminates. In contrast, layerwise thoroughly analyzes individual layers and their interactions, albeit at the cost computational is increased due to the growing number of unknowns directly proportional to the number of layers. Finally, the zigzag theory strikes a balance by incorporating a function into equivalent single-layer kinematics, making it more computationally efficient than layerwise while maintaining precision [14].

Numerous works on zigzag laminated beam theory have been developed. Murakami et al. [6] introduced the 'zigzag' function, focusing on geometric information within a first-order kinematic framework [17]. Tessler et al. [16] advanced the theory by developing a refined zigzag theory, considering geometric and material property influences when determining stress fields. Vidal and Polit [21] refined this approach by incorporating the Murakami zigzag function into sinusoidal higher-order beam kinematics.

Over time, high-order zigzag theories were introduced, coupling these advanced theories with high-order beam kinematics to enhance the accuracy of the results. For instance, in Prado Leite and Rocha [9, 8], a combination of unified beam kinematics and a sinusoidal zigzag function was proposed.

In the present paper, using the Rayleigh-Ritz method, structural responses are analyzed based on the unified zigzag theory for the various boundary conditions, including simple-simple (SS), clamped free (CF), clamped-simple (CS), and clamped-clamped (CC). The results can be used as benchmarks for zigzag high-order theory for the last three because there are few references with these responses for comparison. The structure of this work is as follows: in section 2, the primary geometric characteristics of the problem are defined, and the structural equilibrium is developed mathematically; in section 3, the properties of the problems and their boundary conditions are presented, in addition to the longitudinal shape functions used in each case; in section 4, conclusions are shown that are consistent with the results obtained previously.

2 Mathematical Development

2.1 Geometrical description

Consider a laminated composite beam with length $L = x_b - x_a$, which is subjected to load $q(x)$ and external forces T in the x- and z-directions (Figure 1).

The cross-sectional height is denoted by $2h$, and the thickness of each lamina is $2h^{(k)}$, with $k = 1, 2, ..., N$ represents layer numbering. The depth is denoted as b (Figure 2). The global coordinates of the beam are given by $z_{(i)}$, $i = 0, 1, 2, ..., N$, such that $z_0 = -h$, $z_N = h$, and $z_{(k)} = z_{(k-1)} + 2h^{(k)}$.

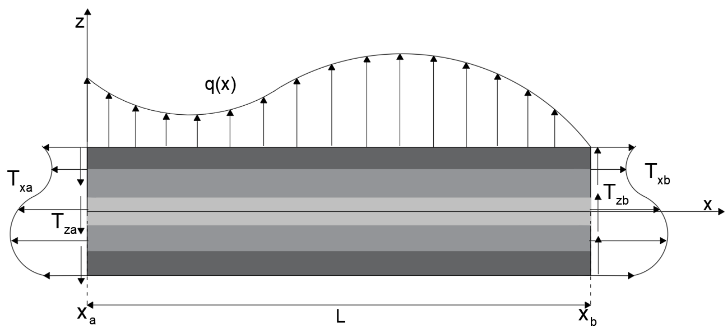

Figure 1: General loading and geometry of a composite laminated beam.

2.2 Kinematics

For all problems discussed, the material have a linear elastic behavior. The displacement fields of various beam theories and zigzag functions are unified with the functions $u^{(k)}(x,z)$ and $w(x,z)$ that are the axial and transverse displacements of each layer, respectively:

$$\begin{aligned} u^{(k)}(x,z) &= u_0(x) - zw_0'(x) + f(z)\phi(x) + \phi_{zz}^{(k)}(z)\psi(x), \\ w(x,z) &= w_0(x). \end{aligned} \tag{2.1}$$

In (Eq. 2.1), $w_0(x)$ is the transverse displacement in midplane of the beam, $u_0(x)$ is the axial displacement in midplane of the beam, $f(z)$ is a vertical shape function that represents the higher-order stress and shear strain distribution (Table 1 shows the principals high order distributions presented on literature), $\phi(x)$ is the shear angle, $\phi_{zz}^{(k)}(z)$ is a generic zigzag function and $\psi(x)$ is a zigzag amplitude function.

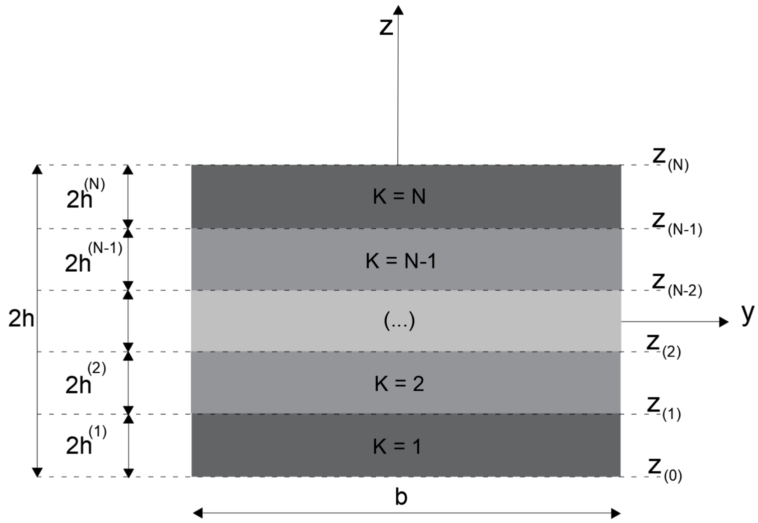

Figure 2: General cross section and layers of a composite laminated beam .

Considering a linear elastic strain behavior, it is possible to calculate the axial and shear strain (Eq. 2.2) from (Eq. 2.1).

$$\begin{aligned} \varepsilon^{(k)}(x,z) &= \frac{\partial u^{(k)}}{\partial x} = u_0'(x) - z w_0''(x) + f(z)\phi'(x) + \phi_{zz}^{(k)}(z)\psi'(x), \\ \gamma^{(k)}(x,z) &= \frac{\partial w}{\partial x} + \frac{\partial u^{(k)}}{\partial z} = f'(z)\phi(x) + \beta^{(k)}(z)\psi(x). \end{aligned} \tag{2.2}$$

Note that $\beta^{(k)}(z)$ is the first derivative of $\phi_{zz}^{(k)}(z)$. The linear constitutive relation (Eq. 2.3) uses the terms $\overline{Q_{11}}^{(k)}$ and $\overline{Q_{55}}^{(k)}$ which are the elastic properties of orthotropic materials described in [22].

$$\begin{aligned} \sigma^{(k)}(x,z) &= \overline{Q_{11}}^{(k)}\varepsilon^{(k)}(x,z) = \overline{Q_{11}}^{(k)}\left[u_0'(x) - z w_0''(x) + f(z)\phi'(x) + \phi_{zz}^{(k)}(z)\psi'(x)\right], \\ \tau^{(k)}(x,z) &= \overline{Q_{55}}^{(k)}\gamma^{(k)}(x,z) = \overline{Q_{55}}^{(k)}\left[f'(z)\phi(x) + \beta^{(k)}(z)\psi(x)\right]. \end{aligned} \tag{2.3}$$

From this unified kinematics, the term $\phi_{zz}^{(k)}(z)$ can be replaced by any zigzag function format, it can be linear or high order. In this paper, the high order sinusoidal format $\phi_{SIN}^{(k)}(z)$ was used [9] (Eq. 2.4).

Table 1: Vertical shape functions for unified high-order beam theory

Source	$f(z)$
Ref. [5]	$\frac{5z}{4}\left[1-\frac{4}{3}\left(\frac{z}{2h}\right)^2\right]$
Ref. [11]	$z\left[1-\frac{4}{3}\left(\frac{z}{2h}\right)^2\right]$
Ref. [18]	$\frac{5h}{\pi}\sin(\frac{\pi z}{2h})$
Ref. [15]	$z\cosh(\frac{z}{2h})-2h\sinh(\frac{z}{2h})$
Ref. [4]	$z\exp\left[-2\left(\frac{z}{2h}\right)^2\right]$
Ref. [1]	$\frac{3\pi}{2}\left[2h\tanh(\frac{z}{2h})-z\sec^2(\frac{z}{2h})\right]$

$$\phi^{(k)}_{SIN}(z)=\sin[\phi^{(k)}_{MUR}(z)]-\left(\frac{z^2}{2z_0}+\frac{2z-3z_0z^2}{12z_N^2}\right)\frac{d\phi^{(0)}_{MUR}(z)}{dz}-\frac{2z^3-3z_0z^2}{12z_N^2}\frac{d\phi^{(N)}_{MUR}(z)}{dz}, \tag{2.4}$$

which is written as a function of the Murakami linear format $\phi^{(k)}_{MUR}(z)$ [6]:

$$\phi^{(k)}_{MUR}(z)=\frac{(-1)^k(z_{(k)}+z_{(k-1)}-2z)}{2h^{(k)}}, \tag{2.5}$$

2.3 Rayleigh-Ritz method

To solve laminated beam problems, the method employed is the Rayleigh-Ritz method utilizing the principle of the first energy theorem. This theorem asserts that, for a conservative system, the equilibrium situation corresponds to a stationary value of the total potential energy , which is equal to zero. The strain energy per unit volume U (internal energy) of the beam is defined as follow:

$$U(\varepsilon) = \sigma(x,z)\varepsilon(x,z). \tag{2.6}$$

Using the differential of a function concept, it is possible write an infinitesimal of this energy (dU):

$$dU = \frac{\partial U}{\partial \varepsilon_x} d\varepsilon_x + \frac{\partial U}{\partial \gamma_{xy}} d\gamma_{xy} = \sigma^{(k)}(x,z)d\varepsilon_x + \tau^{(k)}(x,z)d\gamma_{xy}. \tag{2.7}$$

Continuing the process, (Eq. 2.7) was integrated concerning strain fields, resulting in

$$U = \frac{\overline{Q_{11}^{(k)}}}{2}\left[\varepsilon^{(k)}(x,z)\right]^2 + \frac{\overline{Q_{55}^{(k)}}}{2}\left[\gamma^{(k)}(x,z)\right]^2. \tag{2.8}$$

To obtain the total strain energy U_T, it is necessary to integrate concerning the domain region $dV = dxdA$.

$$\begin{aligned} U_T = {} & A_1\int_0^L [\phi'(x)]^2 dx + A_2\int_0^L \phi'(x)\psi'(x)dx + A_3\int_0^L [\psi'(x)]^2 dx \\ & + A_4\int_0^L \phi'(x)u_0'(x)dx + A_5\int_0^L [u_0'(x)]^2 dx + A_6\int_0^L \psi'(x)u_0'(x)dx \\ & - A_7\int_0^L \phi'(x)w_0''(x)dx - A_8\int_0^L \psi'(x)w_0''(x)dx - A_9\int_0^L u_0'(x)w_0''(x)dx \\ & + A_{10}\int_0^L [w_0''(x)]^2 dx + A_{11}\int_0^L [\phi(x)]^2 dx + A_{12}\int_0^L \phi(x)\psi(x)dx + A_{13}\int_0^L [\psi(x)]^2 dx. \end{aligned} \tag{2.9}$$

The equation (Eq. 2.9) was derived by substituting the strain terms from (Eq. 2.2) and separating the volume integral into two components: one across the cross-sectional area and the other along the x-axis. The integrals related to the cross-sectional area are consolidated into a sum of stiffness terms for each layer, signifying that these integrals are constants for the laminated beam case. These constants are named by A_1 to A_{10}, for stiffness due axial deformation, and A_{11} to A_{13},for stiffness due shear distortion, and are written as $A_n = b\sum_{k=1}^{N}\int_{z_{(k-1)}}^{z_{(k)}} S_{n(k)}(z)dz$, where the terms $S_{n(k)}(z)$ are given in Table 2.

The external energy (Ω) of the beam is given by the (Eq. 2.10),

$$\Omega(F) = Fd, \tag{2.10}$$

where d is the displacement caused by F. In the specific case where the external load depends solely on the x-direction, it is further simplified as

Table 2: Explicit $S_{n(k)}(z)$ terms to describe A_n stiffness

n	$S_{n(k)}$
1	$\frac{Q_{11}^{(k)}}{2}[f(z)]^2$
2	$Q_{11}^{(k)}f(z)\phi_{zz}^{(k)}(z)$
3	$\frac{Q_{11}^{(k)}}{2}[\phi_{zz}^{(k)}(z)]^2$
4	$Q_{11}^{(k)}f(z)$
5	$\frac{Q_{11}^{(k)}}{2}$
6	$Q_{11}^{(k)}\phi_{zz}^{(k)}(z)$
7	$Q_{11}^{(k)}zf(z)$
8	$Q_{11}^{(k)}z\phi_{zz}^{(k)}(z)$
9	$Q_{11}^{(k)}z$
10	$\frac{Q_{11}^{(k)}}{2}z^2$
11	$\frac{Q_{55}^{(k)}}{2}[f'(z)]^2$
12	$Q_{55}^{(k)}f'(z)\beta^{(k)}(z)$
13	$\frac{Q_{55}^{(k)}}{2}[\beta^{(k)}(z)]^2$

$$\Omega = \int_S p(x,y)d(x,y)dS + \int_V b(x,y,z)d(x,y,z)dV = \int_L q(x)w_0(x)dx. \tag{2.11}$$

Hence, the energy functional of the problem is represented by the difference between internal and external energy (Eq. 2.12).

$$\Pi\left[u_0(x), w_0(x), \phi(x), \psi(x)\right] = U_T - \Omega, \tag{2.12}$$

each functional term was approximated using coefficients denoted as $j \times 4$, where j represents the number of approximation terms (Eq. 2.13).

$$\Pi\left[u_{0(j)}, w_{0(j)}, \phi_{(j)}, \psi_{(j)}\right] \ , with \quad j = 1, 2, ..., m. \tag{2.13}$$

In sequence, the first energy theorem is applied when the differential of Π is equal to zero, which means an equilibrium situation for a conservative system (Eq. 2.14 and 2.15).

$$d\Pi\left[u_{0(j)}, w_{0(j)}, \phi_{(j)}, \psi_{(j)}\right] = 0, \tag{2.14}$$

$$\frac{\partial \Pi}{\partial u_{0(j)}} du_{0(j)} + \frac{\partial \Pi}{\partial w_{0(j)}} dw_{0(j)} + \frac{\partial \Pi}{\partial \phi_{(j)}} d\phi_{(j)} + \frac{\partial \Pi}{\partial \psi_{(j)}} d\psi_{(j)} = 0 \ , with \quad j = 1, 2, ..., m. \tag{2.15}$$

To ensure that the $4m$ unknowns, as presented in (Eq. 2.15), are linearly independent and thereby possess a non-trivial solution, it is necessary for the conditions outlined in (Eq. 2.26) to be met, knowing that the differential of the approximating points will never be zero, so that the sum is zero, the partial derivatives of the total energy functional will cancel each other out, thus forming a system of equations.

$$\begin{aligned} du_{0(j)} &\neq 0, \quad \frac{\partial \Pi}{\partial u_{0(j)}} = 0, \\ dw_{0(j)} &\neq 0, \quad \frac{\partial \Pi}{\partial w_{0(j)}} = 0, \\ d\phi_{(j)} &\neq 0, \quad \frac{\partial \Pi}{\partial \phi_{(j)}} = 0, \\ d\psi_{(j)} &\neq 0, \quad \frac{\partial \Pi}{\partial \psi_{(j)}} = 0. \end{aligned} \tag{2.16}$$

3 Results and Discussion

3.1 Problem properties

In (Eq. 3.1), the dimensionless response fields are shown. In this work, the shear stress field was calculated using the equilibrium equations according to the procedure described by Reddy [12].

$$\begin{aligned} {u_a}^{(k)}(x,z) &= u^{(k)}(x,z)\frac{bE_y}{2hq_0}, \qquad w_a(x,z) = w(x,z)\frac{800bh^3E_y}{L^4q_0}, \\ {\sigma_a}^{(k)}(x,z) &= \sigma^{(k)}(x,z)\frac{b}{q_0}, \qquad {\tau_a}^{(k)}(x,z) = \tau^{(k)}(x,z)\frac{b}{q_0}. \end{aligned} \tag{3.1}$$

For the problems analyzed here, the following elastic properties were adopted (laminated beam reinforced with fiber):

$$E_x = 25MPa, \quad E_y = 1MPa, \quad G_{xy} = 0.5MPa, \quad G_{yz} = 0.2MPa, \quad v_{xy} = v_{yz} = 0.25. \tag{3.2}$$

In addition to the response fields presented in (Eq 3.1), the beam slenderness parameter, here called $S = \frac{L}{2h}$, is of great importance for the mechanical analysis of beams. For the boundary conditions SS (simply supported-simply supported), CF (clamped-free), CS (clamped-simply supported), and CC (clamped-clamped), the parameter S was obtained for beams modeled by Reddy kinematics, theory ZZ and using the zigzag function given by (Eq. 2.4). Due to the lack of results in the literature for high-order beams and with boundary conditions other than simply supported, some examples are compared with simpler equivalent single-layer laminated theory, and the differences in results are discussed.

Table 3: Composite laminated beams problems solved

Problem	**Type**	**Boundary conditions**	**Load**	**S**
1	S-S	$w(0) = w(L) = 0$	sinusoidal	4
2	S-S	$w(0) = w(L) = 0$	uniform	5
3	C-F	$u(0) = w(0) = w'(0) = 0$	uniform	5
4	C-S	$u(0) = w(0) = w'(0) = w(L) = 0$	uniform	5
5	C-C	$u(0) = w(0) = w'(0) = u(L) = w(L) = w'(L) = 0$	uniform	5

All five problems described in Table 3 involve laminated beams, each consisting of three layers with a thickness of $h/3$ and stacking sequence 0°/90°/0° (angle between fibers and x-axis). . Furthermore, in Problem 1, the beam supports a sinusoidal load as described in (Eq. 3.3) with a slenderness parameter (S) equal to 4. In problems 2, 3, 4, and 5, the beams support a uniform load defined in (Eq. 3.4) with a slenderness parameter $S = 5$.

$$q(x) = q_0 \sin(\frac{\pi x}{L}). \tag{3.3}$$

$$q\,(x) = q_0. \tag{3.4}$$

3.2 Longitudinal shape functions

Prado Leite and Rocha [10] studied the efficiency of longitudinal shape functions for the Rayleigh-Ritz method applied to zigzag laminated beam problems and concluded the polynomial shape function given by (Eq 3.5) is a satisfactory option for Problems 1 and 2.

$$
\begin{aligned}
w_0(x) &= \sum_{j=1}^{m} w_{0j} x^j (L-x), \qquad u_0(x) = \sum_{j=1}^{m} u_{0j} \frac{d}{dx}\left[x^j (L-x)\right], \\
\phi(x) &= \sum_{j=1}^{m} \phi_j \frac{d}{dx}\left[x^j (L-x)\right], \qquad \psi(x) = \sum_{j=1}^{m} \psi_j \frac{d}{dx}\left[x^j (L-x)\right],
\end{aligned}
\tag{3.5}
$$

To be compatible with the C-F boundary conditions (Problem Type 3), (Eq 3.5) must be changed into the (Eq. 3.6):

$$
\begin{aligned}
w_0(x) &= \sum_{j=1}^{m} w_{0j} x^{(j+1)}, \qquad u_0(x) = \sum_{j=1}^{m} u_{0j} \frac{d}{dx}\left[x^{(j+1)}\right], \\
\phi(x) &= \sum_{j=1}^{m} \phi_j \frac{d}{dx}\left[x^{(j+1)}\right], \qquad \psi(x) = \sum_{j=1}^{m} \psi_j \frac{d}{dx}\left[x^{(j+1)}\right].
\end{aligned}
\tag{3.6}
$$

The longitudinal shape functions must be changed to (Eq. 3.7) when the C-S boundary condition is considered Problem Type 4).

$$
\begin{aligned}
w_0(x) &= \sum_{j=1}^{m} w_{0j} x^{(j+1)} (L-x), \qquad u_0(x) = \sum_{j=1}^{m} u_{0j} \frac{d}{dx}\left[x^{(j+1)} (L-x)\right], \\
\phi(x) &= \sum_{j=1}^{m} \phi_j \frac{d}{dx}\left[x^{(j+1)} (L-x)\right], \qquad \psi(x) = \sum_{j=1}^{m} \psi_j \frac{d}{dx}\left[x^{(j+1)} (L-x)\right].
\end{aligned}
\tag{3.7}
$$

And finally, when the boundary condition is C-C (Problem Type 5), the shape functions in (Eq. 3.8) must be used.

$$
\begin{aligned}
w_0(x) &= \sum_{j=1}^{m} w_{0j} x^{(j+1)} (L-x)^2, \qquad u_0(x) = \sum_{j=1}^{m} u_{0j} \frac{d}{dx}\left[x^{(j+1)} (L-x)^2\right], \\
\phi(x) &= \sum_{j=1}^{m} \phi_j \frac{d}{dx}\left[x^{(j+1)} (L-x)^2\right], \qquad \psi(x) = \sum_{j=1}^{m} \psi_j \frac{d}{dx}\left[x^{(j+1)} (L-x)^2\right].
\end{aligned}
\tag{3.8}
$$

3.3 Main results

For Problem 1, it was previously determined that eight terms (m=8) in the approximate solution series are sufficient to achieve convergence in the displacement and stress fields. The results obtained by Pagano's elasticity theory [7] were adopted as a reference. To compare the accuracy of the results, the work of Kruszewski [5], which adopts the equivalent single-layer theory, was adopted.

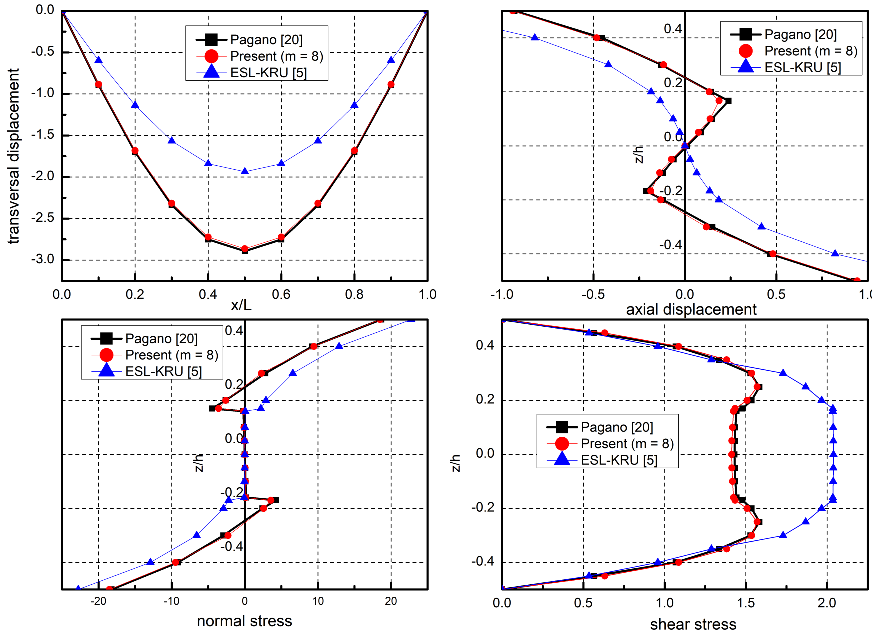

Figure 3: Results of a Problem 1 for (a) transverse displacement $w_a(x, 0)$, (b) axial displacement $u_a(L, z)$, (c) normal stress $\sigma_a(L/2, z)$ and (d) shear stress $\tau_a(0, z)$.

In Figure 3, it is possible to observe that for all response fields, there is a fast convergence approximation ($m = 8$), resulting in good accuracy when compared with Pagano [7]. In addition, the equivalent single-layer formulation, even though using a high order kinematics, shows some expected issues because of the consideration that a laminate is only one layer equivalent with average elastic properties. The equivalent single-layer results cannot capture the zigzag effect in the axial displacement field. Also, the shear and normal stresses do not calculate the interference in inter-laminar regions, problems that the present zigzag theory overcomes because the zigzag functions are inserted into kinematics.

The Problems 2, 3 and 5 (see Table 3) are analyzed and their results are compared with equivalent single-layer formulations presented in [20] and [19]. The maximum transverse displacement values obtained by the present proposal and references [20] and [19] are presented in Table 4. It is worth noting that the formulation proposed in [20] combines Kruszewiski beam kinematics [5] with the ESL theory. The formulation in [19] addresses a quasi-3D model with cubic kinematics for the axial displacement field and quadratic kinematics for the transverse displacement field combined with the

ESL theory.

Table 4: mid-span transversal displacement results.

Ploblem Type	**Thuc et al. [19]**	**Trung-Kien et al. [20]**	**Present**
2	-2.405	-2.412	-2.5945
3	-6.854	-6.813	-7.0835
5	-1.549	-1.536	-1.5936

As shown in Table 3, it is observed that the values obtained in the present formulation agree with the results obtained in [20] and [19], except for the differences already known from the analysis of Problem 1, such as slightly lower values obtained by ESL when compared with ZZ.

Figures 4, 5, and 6 illustrate the displacement fields for Problems 3, 4, and 5, respectively. Since there are no results in the literature for the problems in question via high-order ZZ beam theory, the results are presented using the Rayleigh-Ritz method without dependence on the number of terms in the approximate solution, which will serve as benchmarks for future comparisons.

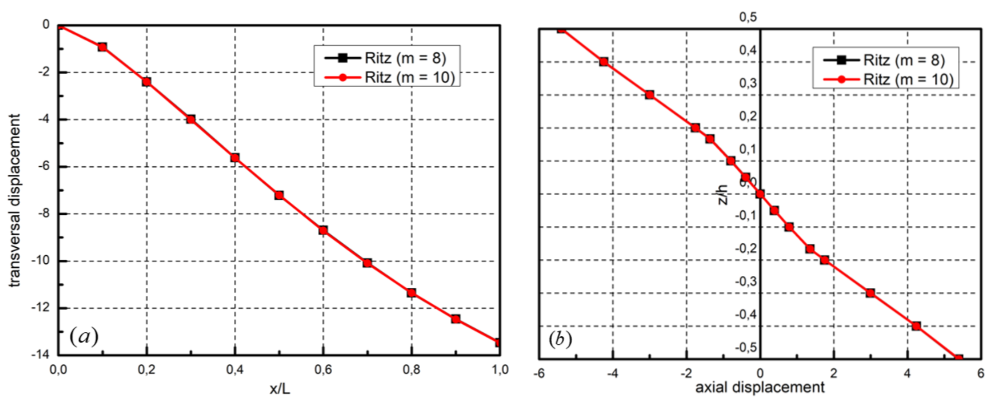

Figure 4: Results of Problem Type 3 (a) transverse displacement $w_a(x, 0)$, (b) axial displacement $u_a(L, z)$.

Regarding Problem 3, which is an isostatic structure, the Rayleigh-Ritz method presents fast convergence, requiring only ten terms to construct the approximate solution. In the case of Problems 4 and 5, with abundant support, 12 terms of the solution series are needed for the results to converge.

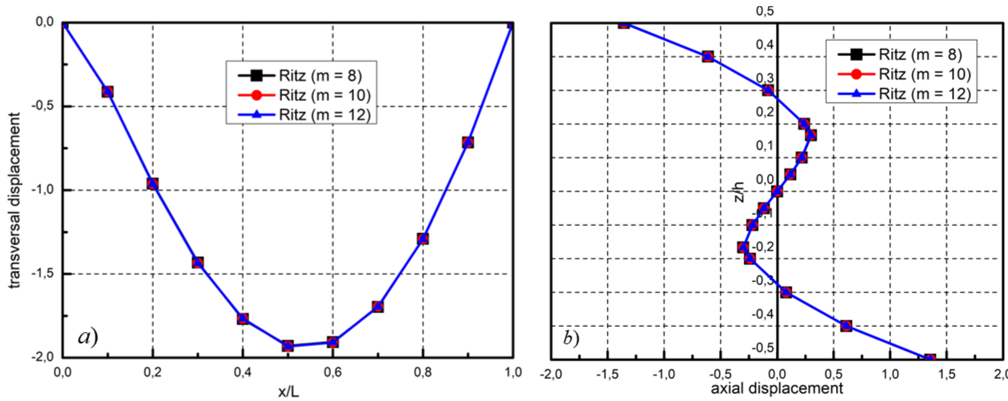

Figure 5: Results of Problem Type 4 (a) transverse displacement $w_a(x, 0)$, (b) axial displacement $u_a(L, z)$.

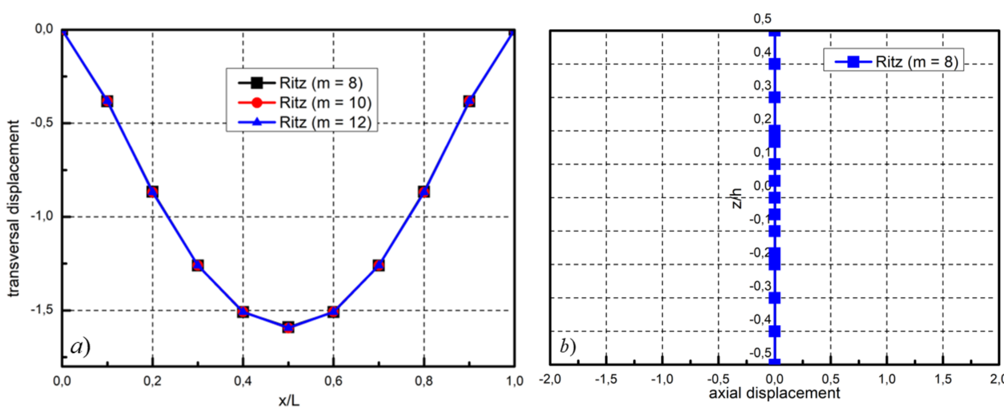

Figure 6: Results of Problem Type 5 (a) transverse displacement $w_a(x, 0)$, (b) axial displacement $u_a(L, z)$.

4 Conclusions

This work developed polynomial functions that consider the boundary conditions in displacements for the supports S-S, C-S, C-F, and C-C. Subsequently, the Rayleigh-Ritz method was applied to the stationary analysis of laminated beams using high-order beam theory combined with the ZZ function proposed in [14]—the results for the problem with S-S boundary conditions agreed with the values in the literature. The results presented for the displacements are compatible with the types of supports C-S, C-F, and C-C. However, for the analysis of the latter, there are no reference values in the literature, and thus, the present work proposes benchmarks for future work.

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