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USING MATRIX DIAGONALIZATION AND PYTHON TO STUDYING A BIDIMENSIONAL LINEAR DISCRETE MIGRATION MODEL

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Abstract

This work was written for students of Mathematics, Biology, Physics, and Engineering. In the field of Linear Algebra, diagonalization is a powerful technique that simplifies matrix calculations by transforming a given matrix into a similar diagonal matrix. This method proves particularly useful for solving systems of linear equations, computing matrix powers, and analyzing the dynamics of linear systems. In this article, we explore a two-dimensional linear discrete model of rural-urban migration. We demonstrate how diagonalization can simplify the model and provide insights into migration patterns. Additionally, we employ Python to develop and analyze this bidimensional linear discrete model. Python's robust computational capabilities allow for complex simulations, which we use to validate the model and visualize migration patterns. The ability to create detailed graphics further aids in interpreting the results. Discrete models, in contrast to continuous ones, offer the distinct advantage of being more accessible for numerical exploration using calculators or computers (in our case, Python). This accessibility enhances the model's pedagogical value, allowing us to effectively demonstrate the connections between mathematical theory and real-world phenomena.

Keywords: Diagonalization; bidimensional Linear discrete dynamical system; real-world application.

Resumo

Este trabalho é destinado a estudantes de Matemática, Biologia, Física e Engenharia. No campo da Álgebra Linear, a diagonalização é uma técnica poderosa que simplifica cálculos matriciais ao transformar uma matriz dada em uma matriz diagonal semelhante.

Este método se mostra particularmente útil para resolver sistemas de equações lineares, calcular potências de matrizes e analisar a dinâmica de sistemas lineares. Neste artigo, exploramos um modelo linear discreto bidimensional de migração rural-urbana. Demonstramos como a diagonalização pode simplificar o modelo e fornecer insights sobre os padrões de migração. Além disso, utilizamos o *software* Python para desenvolver e analisar este modelo linear discreto bidimensional. As robustas capacidades computacionais do Python permitem simulações complexas, que usamos para validar o modelo e visualizar os padrões de migração. A capacidade de criar gráficos detalhados também auxilia na interpretação dos resultados. Modelos discretos, em contraste com os contínuos, oferecem a vantagem distinta de serem mais acessíveis para exploração numérica usando calculadoras ou computadores (no nosso caso, o *software* Python). Essa acessibilidade aumenta o valor pedagógico do modelo, permitindo demonstrar de maneira eficaz as conexões entre a teoria matemática e os fenômenos do mundo real.

Palavras-Chave: Diagonalização; sistema dinâmico linear bidimensional discreto; aplicação no mundo real.

1 Introduction

Mathematical modeling, with its multifaceted nature, is a dynamic process that seamlessly integrates theoretical foundations with practical applications. This fusion inspires learners to deepen their understanding of the world and equips them with the tools to develop strategies for effectively shaping and transforming their environment. The modeling process nurtures critical thinking, strengthens problem-solving skills, and fosters a comprehensive approach to analyzing complex systems. As a result, mathematical modeling serves as a vital bridge between abstract concepts and real-world applications, driving both innovation and informed decision-making.

This perspective aligns with the work of Giordano *et al.* [8] in *A First Course in Mathematical Modeling*, where the authors emphasize the essential integration of theory and practice to demonstrate the application of mathematical concepts across various disciplines. Similarly, Burghes and Borrie [4], along with Bassanezi and Ferreira [1] in *Modelling with Differential Equations* and *Equações Diferenciais Com Aplicações*, respectively, underscore the pivotal role of differential equations in understanding and solving practical problems, thereby reinforcing the theoretical foundation of real-world applications.

To highlight the importance of mathematical modeling in education, it is crucial to demonstrate how mathematics connects with tangible aspects of nature, society, and culture. This connection between the abstract domain of mathematics and real-world contexts is well-recognized in educational literature. Klymchuk and Zverkova [10] stress that employing clear and accessible models is key to forging these connections. Such integration not only enhances students' comprehension of mathematical concepts but also boosts their motivation and equips them to apply mathematical principles effectively. Gainsburg [7] echoes this view, emphasizing that teaching through real-world scenarios deepens understanding and equips students with practical skills for everyday life.

Bassanezi [2], in *Ensino-aprendizagem com modelagem matemática*, emphasizes that modeling in education integrates the creative and exploratory nature of mathematics with its practical applications. This approach allows students to explore diverse pathways for skill development and academic advancement. Similarly, Blum *et al.* (2007), in *Modelling and Applications in Mathematics Education: The 14th ICMI Study*, argue that mathematical modeling cultivates critical thinking and problem-solving abilities, which are essential for addressing complex real-world challenges. This perspective aligns with Stillwell [13], in *Mathematics and Its History*, who illustrates how mathematical concepts have evolved alongside their practical uses, underscoring the enduring importance of modeling in mathematics.

Contributing to this discourse, da Silva [6], in *Mathematics and Its Applications: A Transcendental-Idealist Perspective*, highlights that mathematics' effectiveness is rooted in its intrinsic structure and logical coherence, which harmonize with the empirical nature of the world. Da Silva contends that the remarkable success of mathematics in describing and predicting real-world phenomena arises from a profound congruence between mathematical constructions and human perception. This perspective positions mathematical modeling not merely as a practical tool but as an epistemological bridge linking human cognition to external reality.

Finally, Smith and Morgan [12] underscore the significant impact of a curriculum centered on real-world contexts in mathematics education. Their research demonstrates how such an approach enriches teaching practices and fosters greater student engagement. By connecting theoretical knowledge to practical applications, this method enhances the learning experience and cultivates a deeper appreciation of mathematics' broader relevance.

In a discrete migration model, individuals move between distinct regions or populations at fixed time intervals. These models represent migrations as discrete steps rather than continuous flows, capturing movement patterns occurring each season, year, or over other defined periods. The process is typically described using a *migration matrix*, where each entry represents the probability of an individual transitioning from one region to another within a specific time frame. Each row of the matrix corresponds to all possible destinations for individuals starting in a particular region, while each column indicates all potential origins for those arriving at a given region.

When migration probabilities remain consistent over time, the migration matrix becomes a powerful analytical tool. By *diagonalizing* this matrix, essential characteristics of migration patterns can be revealed, as diagonalization transforms the matrix into a form that simplifies the analysis of long-term system dynamics. This process yields eigenvalues and eigenvectors, which illuminate steady-state distributions, dominant migration pathways, and equilibrium conditions. For instance, eigenvalues close to one may indicate high stability, suggesting that specific populations or regions maintain their sizes over time, while smaller eigenvalues might reflect less stable or infrequent migration trends.

Diagonalization is particularly valuable for studying long-term behavior, as it helps identify stable subspaces and potential steady-state distributions, where population movements stabilize or follow predictable patterns over time. This approach not only clarifies individual migration behaviors but also enables researchers to project future regional population shifts. These insights enhance the understanding of broader demographic trends, highlighting the resilience or vulnerability of specific regions within a migration network.

Advanced mathematical software such as Python significantly enhances the analysis of discrete migration models. This versatile tool offers comprehensive functionalities for managing complex matrix operations, which are fundamental to studying migration dynamics in discrete time. Python excels in tasks such as diagonalization, eigenvalue computation, and matrix transformations, simplifying model equations and uncovering critical patterns in migration data. Additionally, its robust visualization capabilities enable dynamic representations of population flows, helping researchers observe trends, fluctuations, and stability over time. These visualizations not only improve the understanding of current migration dynamics but also provide valuable insights for forecasting future behaviors. By integrating Python's diverse features, researchers can streamline computations, explore various migration scenarios, and develop more accurate predictive models. This approach deepens understanding of migration systems, fostering theoretical advancements and practical applications in population studies.

This article explores the practical implementation of matrix diagonalization techniques, leveraging the computational capabilities of Python to analyze a two-dimensional linear discrete migration model. The objective is to provide a comprehensive understanding of how mathematical tools can be employed to study complex migration dynamics in real-world scenarios. The content is systematically structured to progressively introduce key concepts, methodologies, and computational approaches required to effectively interpret and analyze this migration model.

We begin with the Matrix Diagonalization Theorem, a fundamental concept for understanding the long-term behavior of linear systems. Diagonalization simplifies the computation of matrix powers, a critical step in evaluating the iterative dynamics of migration models.

The next section focuses on the specific migration model under consideration, which categorizes a country's population into rural and urban segments. This model is governed by equations that describe the flow of individuals between these two regions based on their interactions. We will outline the foundational assumptions of the model and discuss how these assumptions influence the system's dynamics over time.

Subsequently, we will perform a mathematical analysis of the model, prioritizing the critical property of solution positivity. Maintaining positivity is essential for ensuring the biological relevance of the model, as it guarantees that population sizes in both rural and urban areas remain non-negative. Negative population values would lack real-world significance and undermine the model's validity. Our analysis will derive the necessary and sufficient conditions for positivity, establishing a robust theoretical foundation for this requirement.

Building on this, we will undertake an analytical examination of the system represented by the matrix equation (3.6). This analysis will focus on the eigenvalues and eigenvectors of the matrix A, which are pivotal in determining the system's long-term behavior. By diagonalizing the matrix, we will gain deeper insights into the stability and steady-state behavior of the migration model, elucidating its dynamics under various scenarios.

Finally, we will demonstrate the application of these theoretical insights using Mathematica. This powerful mathematical software will enable us to simulate the dynamics of the migration model computationally and to explore the impact of varying parameters. Furthermore, Python's visualization tools will enhance our understanding by providing graphical representations of the system's behavior, making complex dynamics more accessible and intuitive.

By the conclusion of this article, readers will acquire a thorough understanding of the mathematical and computational techniques used to analyze migration models. This investigation will not only illuminate key aspects of migration dynamics but also underscore the effectiveness of matrix diagonalization and computational tools like Mathematica in tackling intricate real-world challenges.

2 Preliminaries

In this section, we will provide a brief introduction to some fundamental concepts of dynamical systems and present the Bidimensional Diagonalization Theorem. Throughout this section, our focus will be on the following dynamical system:

$$\begin{pmatrix} p_{n+1} \\ q_{n+1} \end{pmatrix} = A \begin{pmatrix} p_n \\ q_n \end{pmatrix}, \quad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}. \tag{2.1}$$

Definition 2.1. Consider the mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(r) = Ar$. Let $p \in \mathbb{R}^2$. Then:

1. The forward orbit of p is denoted as $\mathcal{O}_f^+(p) = \mathcal{O}^+(p) := \{f^n(p) : n \geq 0\}$.

2. A point p is considered a fixed point of f if $f(p) = p$.

The following result (refer to [9, pp. 126-132]) is pertinent to the analysis of the system (2.1).

Theorem 2.2 (**Bidimensional Diagonalization Theorem**). *Let A be a 2×2 matrix with distinct eigenvalues λ_1 and λ_2, and let $\mathbf{v}_1$ and $\mathbf{v}_2$ be the corresponding eigenvectors. Then, A can be diagonalized as:*

$$A = PDP^{-1},$$

where

$$P = [\mathbf{v}_1 \quad \mathbf{v}_2],$$

with the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ written as column vectors, and

$$D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}.$$

Furthermore, if $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal, then P is an orthogonal matrix, i.e., $P^T = P^{-1}$.

3 Linear Migration Model (LMM)

This model, adapted from [11, p. 144, Section 5.5], examines migration dynamics within a country by dividing the population into rural and urban segments. The model is governed by three primary assumptions:

(a) **Natural Growth:** Both rural and urban populations grow annually at a common rate, denoted by α. This factor represents natural population growth through reproduction. Consequently, in year $n + 1$, the population in each segment will be α times its population in the previous year, n.

(b) **Optimal Rural Base:** Sustaining the total population requires a productive base within the rural segment. This base is defined as a fraction γ of the total population, where γ represents the ideal proportion of the rural population necessary to support overall population growth.

(c) **Rural-to-Urban Migration:** The annual migration rate from rural to urban areas is proportional to the excess rural population relative to the optimal rural base. Specifically, when the rural population exceeds this base, a fraction, defined by the migration factor β, migrates to the urban segment.

Using these assumptions, the model describes the evolution of the population in each segment over time.

Let R_n and U_n denote the rural and urban populations, respectively, in year n. The total population at year n is then $R_n + U_n$. According to assumption (b), the optimal rural base is $\gamma(R_n + U_n)$, and the excess rural population is $R_n - \gamma(R_n + U_n)$. The dynamics of the rural and urban populations are governed by the following state equations:

$$
\begin{aligned}
R_{n+1} &= \alpha R_n - \beta\big(R_n - \gamma(R_n + U_n)\big),\\
U_{n+1} &= \alpha U_n + \beta\big(R_n - \gamma(R_n + U_n)\big),
\end{aligned}
$$

where $\beta > 0$ is the migration factor, and $n \geq 0$. The parameters satisfy the conditions:

$$\alpha > 1, \quad \beta > 0, \quad 0 < \gamma < 1.$$

For simplicity, this system can be rewritten as:

$$
\begin{aligned}
R_{n+1} &= \big(\alpha - \beta(1-\gamma)\big)R_n + \beta\gamma U_n,\\
U_{n+1} &= \beta(1-\gamma)R_n + \big(\alpha - \beta\gamma\big)U_n.
\end{aligned}
\tag{3.1}
$$

Discussion of Parameters and Dynamics: Each parameter in this model plays a distinct role in shaping migration dynamics. Specifically:

1. α **(growth factor):** Represents the natural growth of the population in both segments. Typically, $\alpha > 1$, reflecting annual population increase due to procreation.

2. β **(migration factor):** Regulates the migration rate from rural to urban areas in response to rural population surplus. It determines the yearly adjustment required for the rural population to converge toward the optimal fraction γ.

3. γ **(optimal rural fraction):** Defines the desired proportion of the total population that should remain in rural areas to preserve a balanced demographic structure. A value of γ close to 1 indicates a strong dependence on the rural sector, implying substantial economic reliance on agricultural or rural activities. Conversely, smaller values reflect a greater emphasis on urbanization and urban-based economic growth.

Although these parameters could, in principle, vary over time, they are assumed to remain constant throughout this study. The overall system dynamics and resulting outcomes also depend on the parameter n and the initial conditions, which are analyzed within the theoretical framework introduced in the previous section.

Long-Term Analysis and Interpretation: This model provides a dynamic framework for analyzing population evolution over time. As highlighted in [11, p. 145, Section 5.5], the eigenvalue α indicates that the total population grows at a rate of α. While migration redistributes individuals between rural and urban areas, it does not alter the total population growth rate, which remains driven by α.

Building on the theoretical foundation from the previous section, the relationship between the rural population (R_n) and the urban population (U_n) as functions of n, the model parameters, and the initial conditions is given by:

$$
\begin{aligned}
R_n &= \alpha^n \gamma (R_0 + U_0) + (\alpha - \beta)^n \big[(1-\gamma)R_0 - \gamma U_0\big], \\
U_n &= \alpha^n (1-\gamma)(R_0 + U_0) - (\alpha - \beta)^n \big[(1-\gamma)R_0 - \gamma U_0\big].
\end{aligned}
$$

These equations reveal the system's long-term equilibrium, where population distribution is influenced by parameters γ and β, stabilizing around an optimal rural base. This approach enables scenario-based forecasting and emphasizes the effects of migration on rural and urban populations over time.

3.1 Positivity of the Solution

In this subsection, we examine the positivity property of the model defined in (3.1). Ensuring that the model preserves the non-negativity of initial data throughout its evolution is critical both mathematically and biologically. Positivity, the property by which solutions remain non-negative, is fundamental in models that describe populations, concentrations, or other quantities inherently restricted to non-negative values.

From a biological perspective, positivity ensures that the modeled populations or quantities remain feasible and interpretable. Mathematically, it preserves the integrity and consistency of the model's solutions, particularly in discrete dynamical systems, where iterative updates can amplify errors if positivity is not maintained.

The following theorem establishes a necessary and sufficient condition for the model's positivity. By identifying parameter bounds that ensure positive solutions, we demonstrate that starting from non-negative initial conditions, the quantities R_n and U_n remain non-negative for all $n > 0$. This guarantees both the biological validity and the mathematical robustness of the model over time.

Theorem 3.1. *Let $R_0 \geq 0$ and $U_0 \geq 0$. Then, the inequality*

$$\beta \leq \min\left(\frac{\alpha}{1-\gamma}, \frac{\alpha}{\gamma}\right)$$

is a necessary and sufficient condition to ensure that the solution R_n and U_n of the system (3.1) remains non-negative for all $n \geq 0$.

The proof is structured into two parts: (1) showing sufficiency of the condition for non-negativity, and (2) demonstrating its necessity.

Demonstração. We proceed to analyze the sufficiency and necessity of the stated condition:

Sufficiency: Assume that

$$\beta \leq \min\left(\frac{\alpha}{1-\gamma}, \frac{\alpha}{\gamma}\right).$$

This implies

$$\alpha - \beta(1-\gamma) \geq 0 \quad \text{and} \quad \alpha - \beta\gamma \geq 0. \tag{3.2}$$

To interpret these inequalities, note that $\alpha > 0$, $\beta > 0$, and $0 < \gamma < 1$. This guarantees:

$$\beta\gamma > 0 \quad \text{and} \quad \beta(1-\gamma) > 0. \tag{3.3}$$

Given the initial conditions $R_0 \geq 0$ and $U_0 \geq 0$, the system (3.1) evolves such that R_n and U_n remain non-negative at each step, as the inequalities in (3.2) ensure all contributions to the recurrence relations are non-negative. By induction, $R_n \geq 0$ and $U_n \geq 0$ for all $n \geq 0$, establishing sufficiency.

Necessity: To prove necessity, consider the following specific initial conditions:

1. **Case 1:** Set $R_0 = 1$ and $U_0 = 0$. From system (3.1), the first iteration yields

$$R_1 = \alpha - \beta(1-\gamma) \geq 0,$$

which implies

$$\beta \leq \frac{\alpha}{1-\gamma}. \tag{3.4}$$

2. **Case 2:** Set $R_0 = 0$ and $U_0 = 1$. From (3.1), we find

$$U_1 = \alpha - \beta\gamma \geq 0,$$

which gives

$$\beta \leq \frac{\alpha}{\gamma}. \tag{3.5}$$

By combining (3.4) and (3.5), we obtain

$$\beta \leq \min\left(\frac{\alpha}{1-\gamma}, \frac{\alpha}{\gamma}\right).$$

Thus, this condition is both necessary and sufficient to ensure non-negativity of R_n and U_n for all $n \geq 0$.

□

Theorem 3.1 allows us to derive a simplified condition for the specific case $\gamma = \frac{1}{2}$. This yields the following corollary:

Corollary 3.2. *Let $R_0 \geq 0$ and $U_0 \geq 0$. Then,*

$$\beta \leq 2\alpha$$

is a necessary and sufficient condition to ensure that the solution R_n and U_n of the system (3.1) remains non-negative for all $n \geq 0$.

Demonstração. Substituting $\gamma = \frac{1}{2}$ into the condition from Theorem 3.1, we have

$$\beta \leq \min\left(\frac{\alpha}{1-\frac{1}{2}}, \frac{\alpha}{\frac{1}{2}}\right) = \min(2\alpha, 2\alpha) = 2\alpha.$$

Thus, $\beta \leq 2\alpha$ is both necessary and sufficient to ensure non-negativity of R_n and U_n for all $n \geq 0$, completing the proof. □

3.2 Analytical Examination of the System (3.1)

We consider the system given by Equation (3.1), expressed as:

$$\begin{pmatrix} R_{n+1} \\ U_{n+1} \end{pmatrix} = A \begin{pmatrix} R_n \\ U_n \end{pmatrix} = A^{n+1} \begin{pmatrix} R_0 \\ U_0 \end{pmatrix}, \quad n \geq 0, \tag{3.6}$$

where

$$A = \begin{pmatrix} \alpha - \beta(1-\gamma) & \beta\gamma \\ \beta(1-\gamma) & \alpha - \beta\gamma \end{pmatrix}.$$

Using the notation $(x, y)^T := \begin{pmatrix} x \\ y \end{pmatrix}$, we define $q_n := (R_n, U_n)^T$. Consequently, Equation (3.6) can be rewritten as:

$$q_{n+1} = Aq_n = A^{n+1}q_0.$$

Eigenvalues and Eigenvectors of A: The characteristic equation of A is:

$$\lambda^2 + (-2\alpha + \beta)\lambda + (\alpha^2 - \alpha\beta) = (\lambda - \alpha)(\lambda - \alpha + \beta) = 0.$$

Thus, the eigenvalues of A are:

$$\lambda_1 = \alpha, \quad \lambda_2 = \alpha - \beta. \tag{3.7}$$

The corresponding eigenvectors are:

$$v_1 = \begin{pmatrix} \frac{\gamma}{1-\gamma} \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}. \tag{3.8}$$

From Theorem 2.2, the matrix A can be diagonalized as:

$$P^{-1}AP = J = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha - \beta \end{pmatrix}, \tag{3.9}$$

where

$$P = \begin{pmatrix} \frac{\gamma}{1-\gamma} & -1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad P^{-1} = \begin{pmatrix} 1-\gamma & 1-\gamma \\ \gamma - 1 & \gamma \end{pmatrix}.$$

General Solution of the System: The general solution of Equation (3.6) is:

$$\begin{pmatrix} R_n \\ U_n \end{pmatrix} = A^n \begin{pmatrix} R_0 \\ U_0 \end{pmatrix} = PJ^nP^{-1} \begin{pmatrix} R_0 \\ U_0 \end{pmatrix}.$$

Expanding this expression yields:

$$\begin{pmatrix} R_n \\ U_n \end{pmatrix} = \begin{pmatrix} \frac{\gamma}{1-\gamma} & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \alpha^n & 0 \\ 0 & (\alpha-\beta)^n \end{pmatrix} \begin{pmatrix} 1-\gamma & 1-\gamma \\ \gamma-1 & \gamma \end{pmatrix} \begin{pmatrix} R_0 \\ U_0 \end{pmatrix}.$$

Simplifying further, we obtain:

$$\begin{pmatrix} R_n \\ U_n \end{pmatrix} = \begin{pmatrix} \alpha^n\gamma + (\alpha-\beta)^n(1-\gamma) & \alpha^n\gamma - (\alpha-\beta)^n(1-\gamma) \\ \alpha^n(1-\gamma) - (\alpha-\beta)^n(1-\gamma) & \alpha^n(1-\gamma) - (\alpha-\beta)^n\gamma \end{pmatrix} \begin{pmatrix} R_0 \\ U_0 \end{pmatrix}.$$

From this, the explicit expressions for R_n and U_n are:

$$\begin{aligned} R_n &= \alpha^n\gamma(R_0+U_0) + (\alpha-\beta)^n\big((1-\gamma)R_0 - \gamma U_0\big), \\ U_n &= \alpha^n(1-\gamma)(R_0+U_0) - (\alpha-\beta)^n\big((1-\gamma)R_0 - \gamma U_0\big). \end{aligned} \tag{3.10}$$

Building on the preceding analysis, we can describe the solutions and assess the implications of the model on urban and rural migration patterns. The following deductions are made:

(a) The total population exhibits annual growth at a rate determined by the factor α. This relationship is expressed by the equation:

$$R_n + U_n = \alpha^n(R_0 + U_0),$$

where $n \geq 1$.

(b) When $\lambda_1 > 1$ and $\lambda_2 = \alpha - \beta > 0$ (as derived from (3.10)), the migration pattern is monotonic. This indicates a consistent yearly trend of migration from rural to urban areas. The rural population growth rate remains persistently lower than the total population growth rate, leading to an eventual convergence of the rural-to-total population ratio to zero.

(c) If $\lambda_2 = \alpha - \beta = 0$ (from (3.10)), the relationship between urban and rural populations becomes proportional, with $U_n = \frac{(1-\gamma)}{\gamma} R_n$ for $n \geq 1$. This suggests a stable and optimal balance between the rural and urban populations.

(d) When $\lambda_2 = \alpha - \beta < 0$ (according to (3.10)), the migration pattern takes on an oscillatory nature. Specifically, the migration alternates annually, shifting from rural to urban in one year and reversing to urban to rural in the next.

3.3 Numerical Analysis of the System (3.6)

In this section, we present several phase portraits of the system (3.6) for different values of α and β. It is important to note that the parameters α, β, and γ were not chosen arbitrarily; their values were carefully selected in accordance with Theorem 3.1. This theorem establishes conditions ensuring that the solutions R_n and U_n of the system (3.1) remain non-negative for all $n > 0$, provided that $R_0 \geq 0$ and $U_0 \geq 0$.

Additionally, a phase portrait serves as a visual tool to illustrate the possible trajectories of a system, with arrows indicating the direction of movement through successive iterations of the function. It effectively highlights the geometric features of the orbits. To ensure an accurate depiction, it is essential to carefully select initial points, as they can significantly influence the resulting phase portrait.

3.3.1 Case: $\alpha = \frac{3}{2}, \beta = \frac{1}{2}, \gamma = \frac{1}{3}$

In this subsection, we analyze the system dynamics for the specified parameter values, focusing on the phase portrait construction. The system is transformed into a canonical form to simplify its representation and better illustrate its behavior over time.

We start with the recurrence relation:

$$\begin{pmatrix} R_{n+1} \\ U_{n+1} \end{pmatrix} = A \begin{pmatrix} R_n \\ U_n \end{pmatrix}, \quad \text{where} \quad A = \begin{pmatrix} \frac{7}{6} & \frac{1}{6} \\ \frac{1}{3} & \frac{4}{3} \end{pmatrix}. \tag{3.11}$$

To facilitate the analysis, we apply a nonsingular matrix P to transform A into its Jordan form J. Introducing the variable transformation

$$z_n = P^{-1} q_n \quad \text{or equivalently} \quad q_n = P z_n, \tag{3.12}$$

the system (3.11) becomes

$$z_{n+1} = J z_n, \quad \text{where} \quad z_n = \begin{pmatrix} x_n \\ y_n \end{pmatrix}. \tag{3.13}$$

For the initial condition $q_0 = \begin{pmatrix} R_0 \\ U_0 \end{pmatrix}$, the transformed initial condition is $z_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = P^{-1} \begin{pmatrix} R_0 \\ U_0 \end{pmatrix}$. Notably, the equilibrium properties remain consistent between systems (3.11) and (3.13). Our primary goal is to construct the phase portraits of both systems.

Using (3.11), the eigenvalues of A are determined as $\lambda_1 = \frac{3}{2}$ and $\lambda_2 = 1$, with corresponding eigenvectors:

$$v_1 = \begin{pmatrix} \frac{1}{2} \\ -1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

By Theorem 2.2, the canonical form is given by:

$$P^{-1}AP = J = \begin{pmatrix} \frac{3}{2} & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{where} \quad P = \begin{pmatrix} \frac{1}{2} & -1 \\ 1 & 1 \end{pmatrix}.$$

Phase Portraits: To analyze the transformed system (3.13), the forward orbits are expressed as:

$$\mathcal{O}^+(x_0, y_0) = \{z_n \mid n \in \mathbb{N}\},$$

where the dynamics satisfy:

$$x_{n+1} = \left(\frac{3}{2}\right)^{n+1} x_0 = \frac{3}{2}x_n, \quad y_{n+1} = y_0. \tag{3.14}$$

Figure 1(a) shows the phase portrait of the transformed system $z_{n+1} = Jz_n$. Using (3.14), we compute the first 11 points of two orbits as:

$$\begin{aligned}
\mathcal{O}^+(6,-1) = \{&(6,-1),(9,-1),(13.5,-1),\\
&(20.25,-1),(30.375,-1),(45.5625,-1),\\
&(68.3438,-1),(102.516,-1),(153.773,-1),\\
&(230.66,-1),(345.99,-1)\}, && (3.15)\\
\mathcal{O}^+(6,-2) = \{&(6,-2),(9,-2),(13.5,-2),\\
&(20.25,-2),(30.375,-2),(45.5625,-2),\\
&(68.3438,-2),(102.516,-2),(153.773,-2),\\
&(230.66,-2),(345.99,-2)\}. && (3.16)
\end{aligned}$$

To construct the phase portrait for the original system (3.11), we use the transformation $q_n = Pz_n$ as defined in (3.12). This transformation rotates the axes of the z_n-system into the RU-system. Specifically, the x-axis in the xy-plane maps to a line with slope $\tan(\theta_1) = 2$, and the y-axis maps to a line with slope $\tan(\theta_2) = -1$.

In **Figure 1(b)**, the phase portrait of system (3.11) is shown, with the R-axis aligned along $cv_1 = \begin{pmatrix} \frac{c}{2} \\ c \end{pmatrix}$, and the U-axis along $cv_2 = \begin{pmatrix} -c \\ c \end{pmatrix}$.

Finally, we compute two orbits of the original system:

$$
\begin{aligned}
\mathcal{O}^{+}(4,8) = \{&(4,8),(6,12),(9,18),(13.5,27),(20.25,40.5),\\
&(30.375,60.75),(45.5625,91.125),(68.3438,136.688),\\
&(102.516,205.031),(153.773,307.547),(230.66,461.32)\}.
\end{aligned}
\tag{3.17}
$$

$$
\begin{aligned}
\mathcal{O}^{+}(5,4) = \{&(5,4),(7,6.5),(9.875,10.375),(14.0313,16.3438),(20.0703,25.4922),\\
&(28.8848,39.459),(41.8013,60.7144),(60.7946,92.9789),\\
&(88.8076,141.85),(130.231,215.759),(191.622,327.364)\}.
\end{aligned}
\tag{3.18}
$$

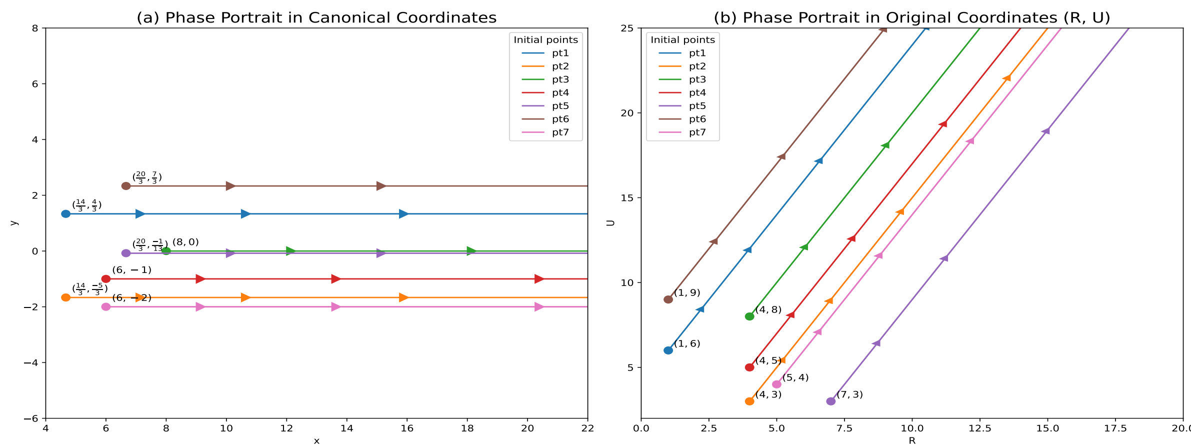

Figura 1: Phase portraits of $z_{n+1} = Jz_n$ (a) and $q_{n+1} = Aq_n$ (b). Each orbit illustrates the first 10 iterates in distinct colors.

3.3.2 Case: $\alpha = \frac{3}{2}, \beta = \frac{1}{4}, \gamma = \frac{1}{3}$

This section presents the phase portrait of the system:

$$
\begin{pmatrix} R_{n+1} \\ U_{n+1} \end{pmatrix} = A \begin{pmatrix} R_n \\ U_n \end{pmatrix}, \quad \text{where} \quad A = \begin{pmatrix} \frac{4}{3} & \frac{1}{12} \\ \frac{1}{6} & \frac{17}{12} \end{pmatrix}. \tag{3.19}
$$

Figure 2(b) illustrates the phase portrait of system (3.19), showing two distinct orbits, each with 10 iterates, represented by different colors.

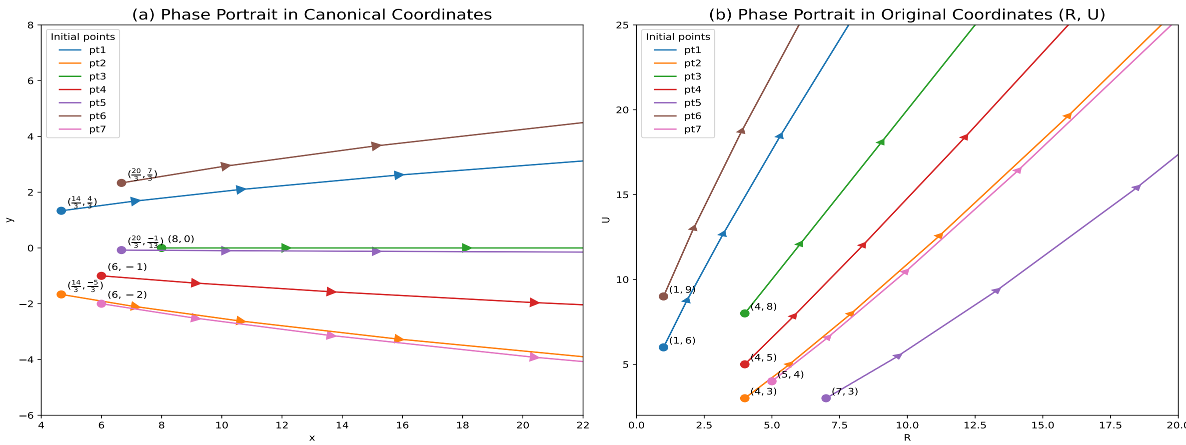

Figura 2: Phase portraits of the systems $z_{n+1} = Jz_n$ and $q_{n+1} = Aq_n$. Each color represents a distinct orbit with 10 iterates.

The following lists detail the first 11 points of two orbits for system (3.19):

$$\begin{aligned}
\mathcal{O}^+(4,8) = \{&(4,8),(6,12),(9,18),(13.5,27),(20.25,40.5),\\
&(30.375,60.75),(45.5625,91.125),(68.3438,136.688),\\
&(102.516,205.031),(153.773,307.547),(230.66,461.32)\}, \qquad (3.20)\\
\mathcal{O}^+(5,4) = \{&(5,4),(7,6.5),(9.875,10.375),(14.0313,16.3438),(20.0703,25.4922),\\
&(28.8848,39.459),(41.8013,60.7144),(60.7946,92.9789),\\
&(88.8076,141.85),(130.231,215.759),(191.622,327.364)\}. \qquad (3.21)
\end{aligned}$$

3.3.3 Case: $\alpha = \frac{3}{2}, \beta = \frac{5}{4}, \gamma = \frac{1}{3}$

In this case, we illustrate the phase portrait of the given system. The system is defined by the following recurrence relation:

$$\begin{pmatrix} R_{n+1} \\ U_{n+1} \end{pmatrix} = A \begin{pmatrix} R_n \\ U_n \end{pmatrix}, \quad \text{where} \quad A = \begin{pmatrix} \frac{2}{3} & \frac{5}{12} \\ \\ \frac{5}{6} & \frac{13}{12} \end{pmatrix}. \tag{3.22}$$

By applying Theorem 2.2, we transform the matrix A into its Jordan canonical form:

$$P^{-1}AP = J = \begin{pmatrix} \frac{3}{2} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}, \quad \text{where} \quad P = \begin{pmatrix} \frac{1}{2} & -1 \\ 1 & 1 \end{pmatrix}.$$

Following the methodology outlined in Section 3.3.1, the phase portrait of the system (3.22) (Fig. 3(b)) is derived from the phase portrait of the simpler system $z_{n+1} = Jz_n$ (Fig. 3(a)).

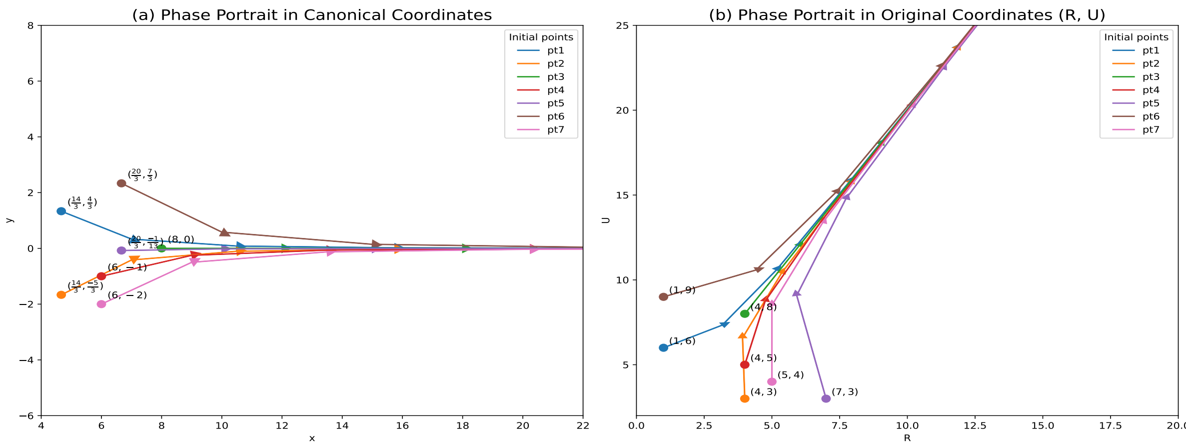

Figura 3: Phase portraits of $z_{n+1} = Jz_n$ (a) and $q_{n+1} = Aq_n$ (b). Each orbit illustrates the first 10 iterates in distinct colors.

To provide further insight, we detail two sample orbits of the system (3.22) below, listing the first 11 elements in each:

$$\begin{aligned}\mathcal{O}^+(4,8) = \{&(4,8),(6,12),(9,18),(13.5,27),(20.25,40.5),\\ &(30.375,60.75),(45.5625,91.125),(68.3438,136.688),\\ &(102.516,205.031),(153.773,307.547),(230.66,461.32)\}.\end{aligned} \tag{3.23}$$

$$\begin{aligned}\mathcal{O}^+(5,4) = \{&(5,4),(5,8.5),(6.875,13.375),(10.1563,20.2188),\\ &(15.1953,30.3672),(22.7832,45.5605),(34.1724,68.3433),\\ &(76.8867,153.773),(115.33,230.6),(172.995,345.99)\}.\end{aligned} \tag{3.24}$$

3.3.4 Case: $\alpha = \frac{3}{2}, \beta = \frac{5}{2}, \gamma = \frac{1}{3}$

In this case, we analyze the phase portrait of the system below.

$$\begin{pmatrix} R_{n+1} \\ U_{n+1} \end{pmatrix} = A \begin{pmatrix} R_n \\ U_n \end{pmatrix}, \quad \text{where} \quad A = \begin{pmatrix} -\frac{1}{6} & \frac{5}{6} \\ \frac{5}{3} & \frac{2}{3} \end{pmatrix}. \tag{3.25}$$

By applying Theorem 2.2, we find the canonical form of matrix A as follows:

$$P^{-1}AP = J = \begin{pmatrix} \frac{3}{2} & 0 \\ 0 & -1 \end{pmatrix}, \quad \text{where} \quad P = \begin{pmatrix} \frac{1}{2} & -1 \\ 1 & 1 \end{pmatrix}.$$

Building on the concepts presented in Section 3.3.1, the phase portrait of the system $z_{n+1} = Jz_n$ (see Fig. 4(a)) allows us to derive the phase portrait of system (3.25), shown in Fig. 4(b).

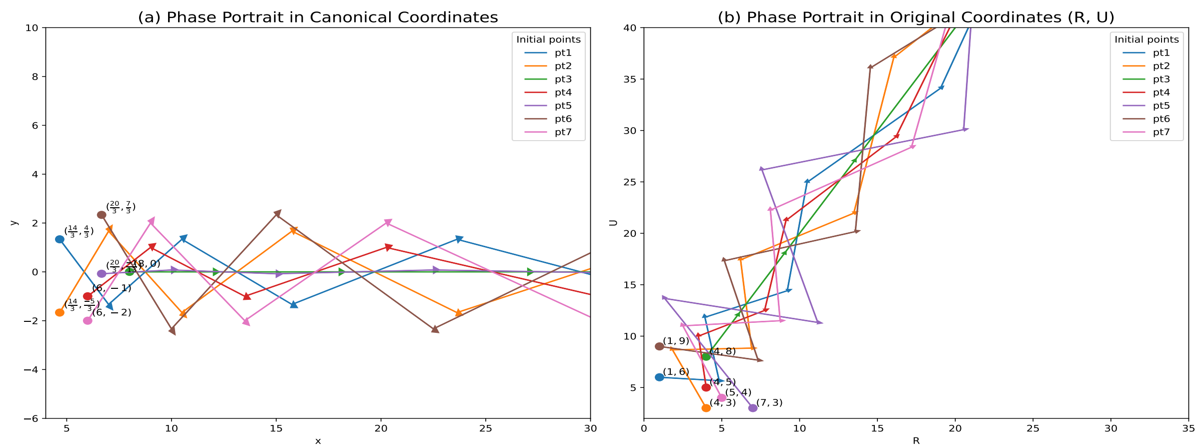

Figura 4: Phase portraits of $z_{n+1} = Jz_n$ (a) and $q_{n+1} = Aq_n$ (b). Each orbit illustrates the first 10 iterates in distinct colors.

Next, we list eleven points for two sample orbits of system (3.25).

$$\begin{aligned}\mathcal{O}^+(4,8) = \{&(4,8), (6,12), (9,18), (13.5,27), (20.25,40.5),\\ &(30.375,60.75), (45.5625,91.125), (68.3438,136.688),\\ &(102.516,205.031), (153.773,307.547), (230.66,461.32)\}.\end{aligned} \tag{3.26}$$

$$\begin{aligned}\mathcal{O}^+(5,4) = \{&(5,4), (2.5,11), (8.75,11.5), (8.125,22.25), (17.1875,28.375),\\ &(20.7813,47.5625), (36.1719,66.3438), (49.2578,104.516),\\ &(78.8867,151.773), (113.33,232.66), (174.995,343.99)\}.\end{aligned} \tag{3.27}$$

3.3.5 Case: $\alpha = \frac{1}{4}, \beta = \frac{3}{2}, \gamma = \frac{1}{3}$

In this case, we describe the phase portrait of the following discrete dynamical system.

$$\begin{pmatrix} M_{n+1} \\ N_{n+1} \end{pmatrix} = A \begin{pmatrix} R_n \\ M_n \end{pmatrix}, \quad \text{where} \quad A = \begin{pmatrix} -\frac{3}{4} & \frac{1}{2} \\ 1 & -\frac{1}{4} \end{pmatrix}. \tag{3.28}$$

Applying Theorem 2.2, we can find the canonical form of A:

$$P^{-1}AP = J = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & -\frac{5}{4} \end{pmatrix}, \quad \text{where} \quad P = \begin{pmatrix} \frac{1}{2} & -1 \\ 1 & 1 \end{pmatrix}.$$

Building on the framework outlined in Section 3.3.1, the phase portrait of the system $z_{n+1} = Jz_n$ (see Fig. 5(a)) allows us to derive the phase portrait of system (3.28), shown in Fig. 5(b).

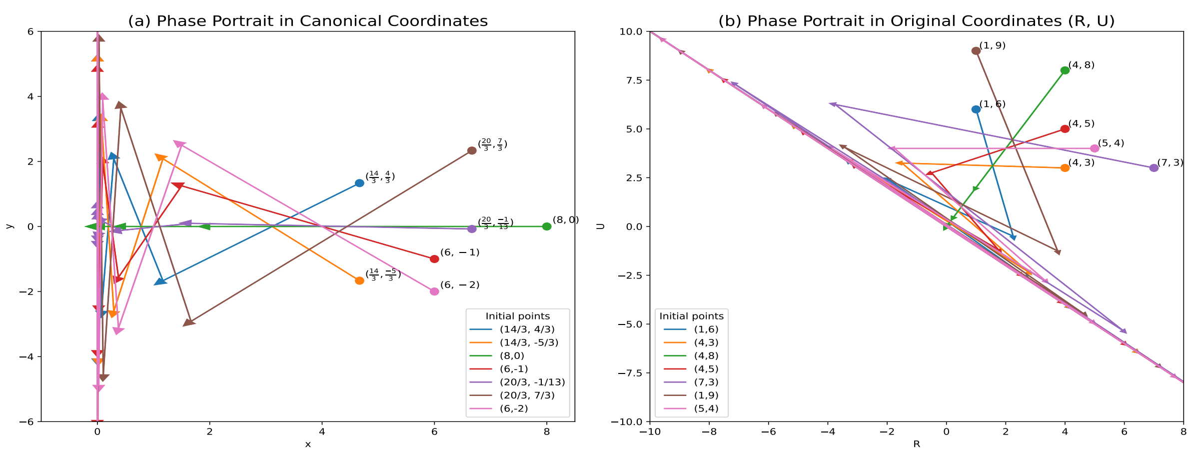

Figura 5: Phase portraits of $z_{n+1} = Jz_n$ (a) and $q_{n+1} = Aq_n$ (b). Each orbit illustrates the first 10 iterates in distinct colors.

Below, we list eleven iterates for two distinct orbits within the system (3.28).

$$\mathcal{O}^+(4,8) = \{(4,8),(2,2),(0,1.5),(0.75,-0.375),(-0.5625,0.84375),$$
$$(0.703125,-0.773438),(-0.738281,0.896484),(0.817383,-0.962402),$$
$$(-0.889893,1.05798),(0.973938,-1.15439),(-1.06416,1.26254)\}. \tag{3.29}$$

$$\begin{aligned}\mathcal{O}^+(5,4) = \{&(5,4),(-0.5,4),(2.25,-1.5),(-1.875,2.625),(2.25,-2.53125),\\ &(-2.39063,2.88281),(2.63672,-3.11133),(-2.87402,3.41455),\\ &(3.14429,-3.72766),(-3.43597,4.0762),(3.75609,-4.45502)\}. \end{aligned} \tag{3.30}$$

Remark 1. In addition, although this does not prevent the publication of the work, it is worth noting a limitation of the adopted linear model. Linear models often produce more predictable and less diverse behaviors compared to nonlinear ones. As a possible future extension, it would be interesting to consider the inclusion of nonlinear terms, such as logistic factors or functions that limit urban growth according to environmental carrying capacity. Another perspective would be to incorporate auxiliary variables, such as infrastructure, income, or public policies, which could enrich the analysis and provide a more comprehensive view of the migration process.

Remark 2. When analyzing system (3.1) in the context of migration, we initially employed a discrete framework. While a continuous approach may better capture the underlying processes, as discussed in [5], discrete models hold significant pedagogical advantages, particularly in educational settings. Their simplicity and computational accessibility make them ideal for engaging students through numerical exploration using calculators or computers.

Discrete models are especially valuable in undergraduate education, serving as an entry point to more advanced mathematical concepts. They bridge the gap between abstract theory and practical application, offering students a hands-on approach to understanding real-world phenomena. By exploring these models, students can see how mathematics is applied across disciplines such as economics, biology, and social sciences, reinforcing the interconnectedness of mathematical modeling and diverse fields.

In addition, discrete models provide a concrete foundation for teaching key mathematical concepts such as matrix operations, eigenvalues, and eigenvectors, which are essential for understanding dynamic systems. Applying these models to realistic scenarios helps students appreciate the practical relevance of mathematics and motivates them to engage with more complex topics. This approach aligns with educational studies that emphasize the importance of linking theoretical knowledge with practical applications. For instance, Giordano *et al.* [8] highlight the role of applied problems in introductory modeling, while Burghes and Borrie [4] and Bassanezi and Ferreira [1] underscore the value of using real-world examples to illustrate theoretical concepts.

Furthermore, the work of Klymchuk and Zverkova [10] emphasizes that accessible and relatable models are crucial for connecting abstract mathematical ideas with tangible experiences. By enabling students to experiment with simulations and numerical solutions, discrete models help demystify mathematics as a purely theoretical discipline and demonstrate its power as a practical tool for understanding and solving real-world problems.

Thus, while a continuous framework may better align with the nature of migration in certain contexts, the educational benefits of discrete models are substantial. They offer students an accessible means of exploring mathematical principles, fostering a deeper appreciation of the role of mathematics in analyzing natural and social phenomena.

4 Summary

This series of works aims to address a common question often raised by students in fundamental linear algebra courses: *How can mathematical concepts be effectively applied across various fields of knowledge?* Gainsburg [7] emphasizes that mathematics acts as a bridge between abstract theories and practical applications, enabling deeper understanding and effective problem-solving. Our exploration highlights the interdisciplinary essence of mathematics and demonstrates its potential in modeling real-world phenomena.

The core focus of this study is a two-dimensional linear discrete rural-urban migration model. This model was chosen for two reasons. First, its simplicity ensures analytical and numerical tractability, making it an accessible educational tool. It allows learners to engage with key mathematical methods, particularly linear algebra and matrix theory. Second, the model's structure offers a practical demonstration of how mathematics can be employed to analyze population dynamics, economics, and sociological trends.

Migration is a phenomenon of great significance, influencing societal structures, economies, and policy-making. By modeling migration flows through matrices, eigenvalues, and eigenvectors, we provide a framework for understanding and analyzing these movements. This illustrates the power of linear algebra and equips students and researchers with tools to tackle real-world challenges.

The simplicity and accessibility of the migration model enable students to explore the interplay between theory and practice in a concrete way. This aligns with the views of Giordano *et al.* [8], Burghes and Borrie [4], and Bassanezi & Ferreira [1], who underscore the importance of linking theoretical foundations to practical applications.

The analytical process involving the migration model shows how diagonalization and eigenvalue analysis can provide insights into long-term behavior. These techniques enable predictions about population dynamics, identification of equilibrium points, and a better understanding of migration patterns. Such analyses are not merely theoretical but inform decision-making in societal and policy contexts.

It is important to note, however, that linear models, such as the one presented here, tend to yield more predictable and less diverse behaviors than nonlinear alternatives. While this simplicity is advantageous for teaching, future research could include nonlinear terms, for example logistic growth or environmental carrying capacity, or auxiliary variables such as infrastructure, income, and public policies. These extensions would enrich the realism of the model and broaden its applicability.

In summary, the study of the two-dimensional migration model transcends exercises in linear algebra. It demonstrates the power of mathematical modeling to illuminate critical aspects of migration dynamics, offering insights for researchers and students alike. By bridging abstract concepts with real-world phenomena, this work underscores the practical impact and relevance of mathematics in addressing complex challenges across disciplines.

5 Appendix

This section presents two Python code snippets for generating phase portraits. Each snippet is named according to the axes it controls. The first snippet, Listing 1, generates a phase portrait using the x- and y-axes, as illustrated in Figure 1(a). The second snippet, Listing 2, produces a phase portrait based on the R- and U-axes, shown in Figure 1(b).

To generate additional phase portraits, modify the entries in the matrices J and A, and adjust the orbit parameters as needed.

Listing 1: Phase Portrait x-y (Figure 1(a))

```
import numpy as np
import matplotlib.pyplot as plt
from fractions import Fraction

# --- Function to generate trajectories ---
def traj(A, pt, t):
trajs = []
for p in pt:
points = [p]
for _ in range(t - 1):
points.append(A @ points[-1])
trajs.append(np.array(points))
return trajs

# --- Function to convert float to fraction string ---
def frac_str(x):
f = Fraction(x).limit_denominator(100)
```

```python
        return f"\\frac{{{f.numerator}}}{{{f.denominator}}}" if f.denominator != 1 else f"{f.numerator}"

        # --- First system ---
        A1 = np.array([[3/2, 0], [0, 1]])
        pt1 = np.array([
        [14/3, 4/3],
        [14/3, -5/3],
        [8, 0],
        [6, -1],
        [20/3, -1/13],
        [20/3, 7/3],
        [6, -2]
        ])
        t1 = 10
        colors1 = plt.cm.tab10.colors
        trajs1 = traj(A1, pt1, t1)

        # --- Create figure ---
        fig, ax = plt.subplots(figsize=(8, 7))
        lines = []
        for i, traj_points in enumerate(trajs1):
        color = colors1[i % len(colors1)]
        line, = ax.plot(traj_points[:, 0], traj_points[:, 1], color=color)
        lines.append(line)
        for j in range(len(traj_points) - 1):
        ax.arrow(traj_points[j, 0], traj_points[j, 1],
        traj_points[j + 1, 0] - traj_points[j, 0],
        traj_points[j + 1, 1] - traj_points[j, 1],
        head_width=0.3, head_length=0.3,
        fc=color, ec=color)
        ax.plot(traj_points[0, 0], traj_points[0, 1], 'o', color=color, markersize=8)
        x_frac = frac_str(traj_points[0, 0])
        y_frac = frac_str(traj_points[0, 1])
        ax.text(traj_points[0, 0] + 0.2, traj_points[0, 1] + 0.2,
        f"$({x_frac}, {y_frac})$", fontsize=10)

        ax.set_xlim(4, 22)
        ax.set_ylim(-6, 8)
        ax.set_xlabel('x')
        ax.set_ylabel('y')
        ax.set_title("(a) Phase Portrait in Canonical Coordinates", fontsize=16)
        ax.legend(lines, [f'pt{i + 1}' for i in range(len(pt1))],
```

```
        title='Initial points')

    plt.tight_layout()
    plt.show()
```

Listing 2: Phase Portrait *R*-*U* (Figure 1(b))

```
import numpy as np
import matplotlib.pyplot as plt
from fractions import Fraction

# --- Function to generate trajectories ---
def traj(A, pt, t):
trajs = []
for p in pt:
points = [p]
for _ in range(t - 1):
points.append(A @ points[-1])
trajs.append(np.array(points))
return trajs

# --- Function to convert float to fraction string ---
def frac_str(x):
f = Fraction(x).limit_denominator(100)
return f"\\frac{{{f.numerator}}}{{{f.denominator}}}" if f.
    denominator != 1 else f"{f.numerator}"

# --- Second system ---
A2 = np.array([[7/6, 1/6], [1/3, 4/3]])
pt2 = np.array([
[1, 6],
[4, 3],
[4, 8],
[4, 5],
[7, 3],
[1, 9],
[5, 4]
])
t2 = 10
colors2 = plt.cm.tab10.colors
trajs2 = traj(A2, pt2, t2)

# --- Create figure ---
fig, ax = plt.subplots(figsize=(8, 7))
lines = []
for i, traj_points in enumerate(trajs2):
```

```
        color = colors2[i % len(colors2)]
        line, = ax.plot(traj_points[:, 0], traj_points[:, 1], color=
            color)
        lines.append(line)
        for j in range(len(traj_points) - 1):
        ax.arrow(traj_points[j, 0], traj_points[j, 1],
        traj_points[j + 1, 0] - traj_points[j, 0],
        traj_points[j + 1, 1] - traj_points[j, 1],
        head_width=0.3, head_length=0.3,
        fc=color, ec=color)
        ax.plot(traj_points[0, 0], traj_points[0, 1], 'o', color=color
            , markersize=8)
        x_frac = frac_str(traj_points[0, 0])
        y_frac = frac_str(traj_points[0, 1])
        ax.text(traj_points[0, 0] + 0.2, traj_points[0, 1] + 0.2,
        f"$({x_frac}, {y_frac})$", fontsize=10)

        ax.set_xlim(0, 20)
        ax.set_ylim(2, 25)
        ax.set_xlabel('R')
        ax.set_ylabel('U')
        ax.set_title("(b) Phase Portrait in Original Coordinates (R, U
            )", fontsize=16)
        ax.legend(lines, [f'pt{i + 1}' for i in range(len(pt2))],
            title='Initial points')

        plt.tight_layout()
        plt.show()
```

Declaration of Conflicts of Interest

The authors declare that there are no potential conflicts of interest associated with this publication.

Referências

[1] Bassanezi, Rodney C.; Ferreira, Wilson: *Equações Diferenciais Com Aplicações*. São Paulo: Harbra Ltda., 1988, 572 p.

[2] Bassanezi, Rodney C.: *Ensino-aprendizagem com modelagem matemática*. São Paulo: Editora Contexto, 2002, 389 p.

[3] Blum, Werner: ICMI Study 14: *Applications and modelling in mathematics education* - Discussion document. ZDM, 34:229–239, 2002.

[4] Burghes, David N.; Borrie, Morag S.: *Modelling with Differential Equations*. Ellis Horwood Series in Mathematics and Its Applications: Statistics and Operational Research. E. Horwood, illustrated edition, reprint, 1981.

[5] Cai, Hai-Ying; Ma, Ning; Junaid, Khan M.: Agent-based model for rural-urban migration: A dynamic consideration. Physica A: Statistical Mechanics and its Applications, 436:806–813, 2015.

[6] da Silva, J. J.: *Mathematics and its Applications*. Synthese Library. Studies in Epistemology, Logic, Methodology, and Philosophy of Science, 385. Springer, Cham, 2017.

[7] Gainsburg, Julie: Real-world connections in secondary mathematics teaching. Journal of Mathematics Teacher Education, 11:199–219, 2008.

[8] Giordano, Frank R.; Weir, Maurice D.: *A First Course in Mathematical Modeling*. Monterey, CA: Brooks/Cole Pub. Co., 1985.

[9] Hirsch, Morris W.; Smale, Stephen: *Differential Equations, Dynamical Systems, and Linear Algebra*. Academic Press, New York-London, 1974. Pure and Applied Mathematics, Vol. 60.

[10] Klymchuk, Sergiy; Zverkova, Tatyana: Role of mathematical modelling and applications in university mathematics service courses: An across countries study. In Matos, João Filipe; Blum, Werner; Houston, Ken; Carreira, Susana Paula, editors, Modelling and Mathematics Education, pages 227–234. Woodhead Publishing, 2001.

[11] Luenberger, David G.: *Introduction to Dynamic Systems: Theory, Models, and Applications*. Wiley, 1st edition, 1979.

[12] Smith, Cathy; Morgan, Candia: Curricular orientations to real-world contexts in mathematics. The Curriculum Journal, 27(1):24–45, 2016.

[13] Stillwell, John: *Mathematics and Its History*. Undergraduate Texts in Mathematics. Springer Science & Business Media, 3rd illustrated edition, 2010.

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