

## MORE OR LESS THAN MONOTONE FUNCTIONS

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### Abstract

The purpose of this article is to present a result on the propagation of the monotonicity of a function, between topological spaces, from a dense subset of its range to the whole range. We show that the propagation occurs provided that the domain is pre-normal, the function is open-and-closed and the preimage of each point is closed. No assumption is superfluous.

**Keywords:** Monotonicity, connectedness, open-and-closed function.

## 1 Introduction and results

A function  $f : X \rightarrow Y$ , between topological spaces, is said to be:

- *monotone* provided for each  $y \in Y$  the preimage  $f^{-1}(y)$  is connected.
- *c-monotone* provided it is surjective and for each connected subset  $C \subset Y$  the preimage  $f^{-1}(C)$  is connected.
- *d-monotone* provided it is surjective and there exists a dense subset  $B \subset Y$  such that for each  $b \in B$  the preimage  $f^{-1}(b)$  is connected.

The concept of *monotonicity* presented here rescues the usual one for self-functions of the real line. The same terminology is adopted in [2, p.358]. However, that book requires the map  $f$  to be continuous. Here we leave the continuity hypothesis for when it is necessary, namely, in Theorem 1.1.

The reader will not find the terms *c-monotone* and *d-monotone* in the regular literature. These terms have been coined here for the sake of simplicity, i.e. they are just auxiliary terminologies.

Understanding the connectedness of the preimage of connected sets can be relevant in several contexts, for instance, in foliation theory, global injectivity problems and studies related to the famous Jacobian Conjecture; see [1, 3, 4].

For the notions of *c-monotonicity* and *d-monotonicity* we require that the function is surjective, because without this assumption the notions lose their meaning. In fact, there exist non-surjective functions  $f : X \rightarrow Y$  for which:

- There exists a connected subset  $C \subset Y$  such that  $C \cap f(X)$  is disconnected; it is not reasonable to consider such a subset to deny the c-monotonicity. For instance, let  $f : \mathbb{Q} \rightarrow \mathbb{R}^2$  be the function given by  $f(x) = (x, 0)$ , and consider the connected subset  $C = \{(\cos t, \sin t) : t \in [0, \pi]\} \subset \mathbb{R}^2$ . The intersection  $C \cap f(\mathbb{Q}) = \{(-1, 0), (1, 0)\}$  is disconnected, as well as its preimage  $f^{-1}(C) = \{-1, 1\}$ . On the other hand, each connected subset of  $f(\mathbb{Q}) = \mathbb{Q} \times \{0\}$  has connected preimage; in fact, such subsets are the one-point sets. Therefore, the surjective map  $f|_1 : \mathbb{Q} \rightarrow f(\mathbb{Q})$ , obtained from  $f$  by restricting the range to the image  $\text{im}(f)$ , is c-monotone.
- There exists a dense subset  $B \subset Y$  such that  $B \cap f(X)$  is empty; it is not reasonable to consider such a subset  $B$  as proving d-monotonicity. For instance, let  $f : \mathbb{Q} \times \mathbb{Q} \rightarrow \mathbb{R}$  be the function given by  $f(x, y) = x$ . The subset  $B = \mathbb{R} \setminus \mathbb{Q}$  (of the irrational numbers) is dense in  $\mathbb{R}$ , and the preimage  $f^{-1}(b)$  is connected for each  $b \in B$  (by vacuity). However,  $f^{-1}(x) = \{x\} \times \mathbb{Q}$  is disconnected for each  $x \in \mathbb{Q} = \text{im}(f)$ . Therefore, the surjective map  $f|_1 : \mathbb{Q} \times \mathbb{Q} \rightarrow \text{im}(f)$ , obtained from  $f$  by restricting the range to its image, is not d-monotone.

The following result corresponds to Theorem 6.1.29 of [2].

**Theorem 1.1.** *A continuous and surjective monotone function that is either open or closed is c-monotone.*

The next result gives non-superfluous conditions under which d-monotonicity forces monotonicity.

**Theorem 1.2.** *Let  $X$  be a pre-normal space and let  $f : X \rightarrow Y$  be an open-and-closed function such that for each  $y \in Y$  the preimage  $f^{-1}(y)$  is closed. If  $f$  is d-monotone then  $f$  is monotone. All the assumptions are essential.*

In the statement of Theorem 1.2, we use the following terminology: a topological space  $X$  is called *pre-normal* provided for each pair of disjoint closed subsets  $A, B \subset X$  there exist disjoint open subsets  $U, V \subset X$  such that  $A \subset U$  and  $B \subset V$ .

This terminology is unusual; however, it adapts well to the usual: according to [5], a topological space  $X$  is *normal* if it is *pre-normal* and its one-point subsets are closed, i.e. *normal* = *pre-normal* +  $T_1$  axiom.

Theorem 1.2 is a direct consequence of the following one:

**Theorem 1.3.** *Let  $X$  be a pre-normal space and let  $f : X \rightarrow Y$  be an open-and-closed function. Let  $B \subset Y$  be a subset with the property that for each  $b \in B$  the preimage  $f^{-1}(b)$  is connected. If  $y_0$  is a point in the closure of  $B$  whose preimage  $f^{-1}(y_0)$  is closed then  $f^{-1}(y_0)$  is connected. All the assumptions are essential.*

It follows from Theorem 1.3 that, under the assumptions of Theorem 1.2, the set of the points in  $Y$  with connected preimage is closed.

Note that no hypothesis about the topological space  $Y$  or the continuity of the function  $f$  is assumed in Theorems 1.2 and 1.3.

## 2 Proofs

As we have said, Theorem 1.2 is a direct consequence of Theorem 1.3. So, we just need to prove the latter.

**Lemma 2.1.** *Let  $f : X \rightarrow Y$  be a closed function. For each open subset  $A \subset X$ , the subset  $A_f := \{y \in Y : f^{-1}(y) \subset A\}$  is open in  $Y$ .*

*Proof.* Since  $A$  is open and  $f$  is closed, the subset  $f(X \setminus A)$  is closed in  $Y$ . The result follows by proving that  $A_f = Y \setminus f(X \setminus A)$ , which is straightforward.  $\square$

*Proof of Theorem 1.3.* Suppose that  $f^{-1}(y_0)$  is disconnected, we say  $f^{-1}(y_0) = C_1 \cup C_2$  with  $C_1$  and  $C_2$  disjoint nonempty closed sets in  $f^{-1}(y_0)$ . Since  $f^{-1}(y_0)$  is closed in  $X$ , also  $C_1$  and  $C_2$  are closed in  $X$ . Since  $X$  is pre-normal, there exist disjoint open sets  $A_1, A_2 \subset X$  such that  $C_1 \subset A_1$  and  $C_2 \subset A_2$ . Since  $f$  is open, both  $f(A_1)$  and  $f(A_2)$  are open in  $Y$ . Consider the open set  $A = A_1 \cup A_2$  in  $X$ . Since  $f$  is closed, it follows from Lemma 2.1 that  $A_f$  is open in  $Y$ . Hence, the intersection  $V = f(A_1) \cap f(A_2) \cap A_f$  is an open neighborhood of  $y_0$  in  $Y$ . Since  $y_0$  belongs to the closure of  $B$ , there exists a point  $b \in B \cap V$ . From the fact that  $b \in A_f$  follows that  $f^{-1}(b) \subset A = A_1 \cup A_2$ . Since, moreover,  $b \in f(A_1) \cap f(A_2)$ , the preimage  $f^{-1}(b)$  intersects both  $A_1$  and  $A_2$ . This implies that  $f^{-1}(b)$  is disconnected, which is in contradiction with  $b \in B$ .  $\square$

Next, we prove that each assumption in the statement of Theorem 1.3 (and thus of theorem 1.2) is essential, i.e. the result fails in the absence of each individual assumption (if all other assumptions are maintained).

- *The assumption “ $X$  is pre-normal” is essential.*

Consider  $X = \{a, b, c, x_1, x_2\}$  and  $Y = \{d, y_0\}$  endowed, respectively, with the topologies  $\tau_X$  generated by the sub-base  $\mathcal{B}_X = \{\{a, b, c\}, \{a, b, x_1\}, \{a, c, x_2\}\}$  and  $\tau_Y = \{\emptyset, \{d\}, Y\}$ . The space  $X$  is not pre-normal; in fact, the subsets  $\{x_1\}$  and  $\{x_2\}$  are closed but they are not contained in disjoint open sets (each neighborhood of  $x_1$  contains  $a$ , as well as each neighborhood of  $x_2$ ). Consider the (continuous) surjective function  $f : X \rightarrow Y$  defined by  $f(a) = f(b) = f(c) = d$  and  $f(x_1) = f(x_2) = y_0$ . This function is open-and-closed. The point  $y_0$  belongs to the closure of  $B = \{d\} \subset Y$ , and its preimage  $f^{-1}(y_0) = \{x_1, x_2\}$  is closed. Moreover,  $f^{-1}(d) = \{a, b, c\}$  is connected, since  $a$  belongs

to each nonempty open subset of  $X$  and, in particular, to each open subset of the subspace  $f^{-1}(d)$ . However,  $f^{-1}(y_0) = \{x_1, x_2\}$  is disconnected; in fact, the relative open sets  $\{x_1\} = f^{-1}(y_0) \cap \{a, b, x_1\}$  and  $\{x_2\} = f^{-1}(y_0) \cap \{a, c, x_2\}$  provides a separation for  $f^{-1}(y_0)$ .

- *The assumption “ $f$  is open” is essential.*

Consider the circle  $S^1$  as a subset of the complex plane. Of course,  $S^1$  is normal and so pre-normal. Consider the (continuous) surjective function  $f : [0, 2\pi] \rightarrow S^1$  given by  $f(x) = e^{ix}$ . This function is closed but not open. For each point  $b$  in the dense subset  $B = S^1 \setminus \{1\} \subset S^1$ , the preimage  $f^{-1}(b)$  is closed and connected (it consists of a single point), but the preimage  $f^{-1}(1) = \{0, 2\pi\}$  is closed and disconnected.

- *The assumption “ $f$  is closed” is essential.*

Consider the punctured plane  $X = \mathbb{R}^2 \setminus \{(0, 0)\}$ , which is normal and so pre-normal, and the (continuous) surjective function  $f : X \rightarrow \mathbb{R}$  given by  $f(x, y) = x$ . This function is open but not closed. For each point  $b$  in the dense subset  $B = \mathbb{R} \setminus \{0\} \subset \mathbb{R}$ , the preimage  $f^{-1}(b) = \{b\} \times \mathbb{R}$  is closed and connected, but  $f^{-1}(0) = \{0\} \times B$  is closed and disconnected.

- *The assumption “ $f^{-1}(y_0)$  is closed” is essential.*

Consider  $X = \{a, b, x_1, x_2\}$  and  $Y = \{c, d, y_0\}$  endowed with the topologies

$$\begin{aligned}\tau_X &= \{\emptyset, \{a\}, \{a, x_1\}, \{a, x_2\}, \{a, x_1, x_2\}, X\} \text{ and} \\ \tau_Y &= \{\emptyset, \{c\}, \{c, y_0\}, Y\}.\end{aligned}$$

The space  $X$  is pre-normal by vacuity (there is not a pair of disjoint closed subsets in  $X$ ). Let  $f : X \rightarrow Y$  be the function given by  $f(a) = c$ ,  $f(b) = d$  and  $f(x_1) = f(x_2) = y_0$ . This function is (continuous) surjective and open-and-closed. The subset  $\{c\} \subset Y$  is dense in  $Y$ . Moreover,  $f^{-1}(c) = \{a\}$  is connected. Also  $f^{-1}(d) = \{b\}$  is connected, and this fact is in harmony with Theorem 1.3, since the subset  $\{b\}$  is closed in  $X$ . On the other hand,  $f^{-1}(y_0) = \{x_1, x_2\}$  is not closed and, because of this, Theorem 1.3 does not work for this preimage. Indeed,  $f^{-1}(y_0) = \{x_1, x_2\}$  is disconnected; in fact, the relative open subsets  $\{x_1\} = \{a, x_1\} \cap f^{-1}(y_0)$  and  $\{x_2\} = \{a, x_2\} \cap f^{-1}(y_0)$  provide a separation for  $f^{-1}(y_0)$ .

In the four examples just presented, the function  $f$  is continuous. This shows that none of the hypotheses of Theorems 1.2 and 1.3 can be replaced by the continuity of the function.

## References

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Received 15 January 2024

Accepted 04 April 2024