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PARTITIONS, TWO-LINE MATRICES AND T-SQUARED PARTITIONS

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Dedicated to Professor J. P. O. Santos on occasion of his 73rd birthday.

Abstract

A natural number m is said to admit a *t-squared partition* if we can find $c_1, \ldots, c_t \in \mathbb{N}$ such that

$$m = (c_1 + c_2 + \cdots + c_t)^2 \; + \; 2(c_1^2 + c_2^2 + \cdots + c_t^2).$$

In this paper, we present a complete characterization of integers that admit t-squared partitions, and we will also introduce a correspondence between the number of partitions of n, both with and without constraints, and the number of representations of integers $m \in (1, n^2)$ as t-squared partitions.

Keywords: Partitions, Matrix Representation, Partitions Identities

1 Introduction

In 1900, Frobenius[4] published a paper introducing a connection between partitions and two-line matrices. In 1984, Andrews[1] revisited these ideas, demonstrating a relationship between these matrix representations and Elliptic Theta functions. A new correspondence between partitions and two-line matrices was introduced by Mondek, Ribeiro, and Santos[8], with an important feature being that the conjugate of a partition can also be obtained from its corresponding matrix. This theory was further developed in the works of Brietzke, Santos, and Silva[2, 3], where generalizations involving Mock Theta Functions are presented. In 2018, Matte and Santos[9] presented an intriguing correspondence between partitions of n, two-line matrices, paths in the Cartesian plane, and integers $m \in (1, n^2)$, which admit a partition into distinct odd parts greater than one. The description of this correspondence is known as the *Path Procedure*. In that paper, Matte and Santos[9] studied these partitions in detail, deriving interesting properties.

Motivated by these ideas, Santos and I introduced the concept of t-squared partitions, as presented in Godinho and Santos[5, 6], where we showed that all integers $m \in (1, n^2)$ admitting partitions into distinct odd parts greater than one, as mentioned in Matte and Santos[9], also admit t-squared partitions. Building on this concept, we presented new correspondences between partitions, both with and without constraints, and the number of representations of an integer $m \in (1, n^2)$ as t-squared partitions.

This article has an expository nature and aims to present the ideas described in Godinho and Santos[5, 6] in a unified manner, with the expectation that this account will spark new interest in the subject and potentially catalyze further innovative developments.

2 Two-Line Matrices

Let us start by introducing the correspondence between partitions and matrices, recalling that a partition of a positive integer n is a non-decreasing sequence of natural numbers whose sum is equal to n.

Let $n, \beta, \delta \in \mathbb{N} \cup \{0\}$, with $n \geq 1$, $n > \beta$ and define $\mathbb{M}(n, \beta, \delta)$ to be the set of all two-line matrices

$$M = \left(\begin{array}{cccc} a_1 & a_2 & \cdots & a_s \\ b_1 & b_2 & \cdots & b_s \end{array} \right), \tag{2.1}$$

such that $a_j, b_j \in \mathbb{N} \cup \{0\}$, $(a_j, b_j) \neq (0, 0)$, $1 \leq j \leq s$, and

$$a_s = \beta, \quad a_j = a_{j+1} + b_{j+1} + \delta \quad \text{and} \quad \sum_{i=1}^{s} (a_i + b_i) = n. \tag{2.2}$$

Let us also define

$$\ell(M) = (a_1 + b_1) + \cdots + (a_s + b_s) = n. \tag{2.3}$$

Lemma 2.1. *Let $M \in \mathbb{M}(n, \beta, \delta)$, written as in (2.1), then*

(*i*) $a_{s-1} \geq \beta + \delta$, *and* $a_j \geq a_{j+1} + \delta$, *for* $1 \leq j \leq s-2$;

(*ii*) $a_{s-j} = a_s + (b_s + \cdots + b_{s-j+1}) + j\delta$, *for* $j = 1, \ldots s-1$;

(*iii*) $\ell(M) = (a_1 + b_1) + (\sum_{j=1}^{s-1} a_j) - (s-1)\delta$;

Proof. The first two statements follow directly from (2.2). It follows from item (ii) that $a_1 = a_s + \sum_{j=2}^{s} b_j + (s-1)\delta$, hence

$$\ell(M) = \sum_{j=1}^{s}(a_j + b_j) = (a_1 + b_1) + \sum_{j=1}^{s-1} a_j - (s-1)\delta.$$

□

Remark 2.2. *Observe that if $i \neq j$ then $\mathbb{M}(i, \beta, \delta) \cap \mathbb{M}(j, \beta, \delta) = \emptyset$, otherwise we would have a matrix M such that $\ell(M) = i$ and $\ell(M) = j$. Besides that, if we denote by $\mathbb{M}^*(n, \beta, \delta)$ the subset of $\mathbb{M}(n, \beta, \delta)$ of all matrices with at least two columns then*

$$\mathbb{M}(n, \beta, \delta) = \mathbb{M}^*(n, \beta, \delta) \cup \left\{ \begin{pmatrix} \beta \\ n-\beta \end{pmatrix} \right\}. \tag{2.4}$$

Definition 2.3. Let $n \in \mathbb{N}$, $n \geq 2$ and $\beta, \delta \in \mathbb{N} \cup \{0\}$, with $n > \beta$. Define

$$\mathbb{M}_0(n, \beta, \delta) = \{M \in \mathbb{M}(n, \beta, \delta) \mid b_1 = 0\}. \tag{2.5}$$

Lemma 2.4. *Let $n \in \mathbb{N}$ and $\beta, \delta \in \mathbb{N} \cup \{0\}$, $n > \beta$. Then we have*

$$|\,\mathbb{M}(n, \beta, \delta)\,| \;=\; \sum_{j=2}^{n} |\,\mathbb{M}_0(j, \beta, \delta)\,| + 1.$$

Proof. Since we are assuming $n > \beta = a_s$, it follows from Remark 2.2 that

$$\begin{pmatrix} \beta \\ n-\beta \end{pmatrix} \notin \mathbb{M}_0(j, \beta, \delta),$$

and that

$$\bigcup_{j=2}^{n} \mathbb{M}_0(j, \beta, \delta)$$

is a disjoint union. In order to complete this proof we present the following 1-1 correspondence between $\mathbb{M}^*(n, \beta, \delta)$ (see (2.4)) and the disjoint union $\bigcup_{j=2}^{n} \mathbb{M}_0(j, \beta, \delta)$:

$$\begin{array}{ccc} M & \longleftrightarrow & M_0 \\ \begin{pmatrix} a_1 & a_2 & \cdots & a_{s-1} & a_s \\ b_1 & b_2 & \cdots & b_{s-1} & b_s \end{pmatrix} & \longleftrightarrow & \begin{pmatrix} a_1 & a_2 & \cdots & a_{s-1} & a_s \\ 0 & b_2 & \cdots & b_{s-1} & b_s \end{pmatrix}. \end{array} \tag{2.6}$$

Since $\ell(M) = n$ then $\ell(M_0) = n - b_1$, hence $M_0 \in \bigcup_{j=2}^{n} \mathbb{M}_0(j, \beta, \delta)$. On the other hand given any $M_0 \in \bigcup_{j=2}^{n} \mathbb{M}_0(j, \beta, \delta)$, we can find $b_1 \in \mathbb{N} \cup \{0\}$, such that $\ell(M_0) + b_1 = n$, and determine the matrix $M \in \mathbb{M}^*(n, \beta, \delta)$ (see (2.6)). Now the result follows from the fact that $\mathbb{M}^*(n, \beta, \delta) = \mathbb{M}(n, \beta, \delta) - 1$, according to (2.4). □

2.1 Partitions and two-line Matrices

Given a matrix $M \in \mathbb{M}(n, \beta, \delta)$, written as (2.1), if we define $\mu_j = a_j + b_j$, then we have

$$n = \mu_1 + \cdots + \mu_s,$$

where $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_s \geq \beta$ and $\mu_j - \mu_{j-1} \geq \delta$ (see (2.2)). Thus we have a partition of n with the smallest part being at least β and the minimum distance between parts being at least δ.

On the other hand, given a partition $n = \mu_1 + \cdots + \mu_s$, with $\mu_s \geq \beta$ and $\mu_{j-1} - \mu_j \geq \delta$, we can write

$$\begin{array}{rcl} \mu_s &=& \beta + b_s &=& a_s + b_s \\ \mu_{s-1} &=& (\mu_s + \delta) + b_{s-1} &=& a_{s-1} + b_{s-1}, \\ \mu_{s-2} &=& (\mu_{s-1} + \delta) + b_{s-2} &=& (a_{s-1} + b_{s-1} + \delta) + b_{s-2} &=& a_{s-2} + b_{s-2}, \end{array}$$

and continuing this process we obtain a matrix $M \in \mathbb{M}(n, \beta, \delta)$ (see (2.1)). This establish a bijection between the set $\mathbb{M}(n, \beta, \delta)$ and the set of all partitions of n with the smallest part being at least β and the minimum distance between parts being at least δ.

In particular we have that

(a) The number of unrestricted partitions of n is equal to the cardinality of $\mathbb{M}(n, 1, 0)$;

(b) The number of partitions of n into distinct parts is equal to the cardinality of $\mathbb{M}(n, 1, 1)$;

(c) The number of partitions of n where the difference between two parts is at least two (Rogers-Ramanujan of type I) is equal to the cardinality of $\mathbb{M}(n, 1, 2)$;

(d) The number of partitions of n where the difference between two parts is at least two and each part is greater than one (Rogers-Ramanujan of type II) is equal to the cardinality of $\mathbb{M}(n, 2, 2)$.

3 t-Squared Partitions

We say that $m \in \mathbb{N}$ admits a *t-squared partition* if we can find $c_1, \ldots, c_t \in \mathbb{N}$ such that

$$m = (c_1 + c_2 + \cdots + c_t)^2 \; + \; 2(c_1^2 + c_2^2 + \cdots + c_t^2). \tag{3.1}$$

For example, the numbers 107 and 144 can be written as

$$\begin{aligned} 107 &= (5+2)^2 \quad + \quad 2 \times (5^2+2^2) \\ 144 &= (3+3+1+1+1+1)^2 \quad + \quad 2 \times (3^2+3^2+1^2+1^2+1^2+1^2) \end{aligned}$$

that is, 107 admits a 2-squared partition and 144 admits a 6-squared partition.

As mentioned above, the final product of the Path Procedure consists of integers $m \in (1, n^2)$ that admit a partition into distinct odd parts greater than one. In Godinho-Santos[5], it is proved that these integers m also admit t-squared partitions. In this section, we give a complete characterization of these special integers, proving that m admits a t-squared partition if and only if $m \equiv 0$ or $3 \pmod 4$, with the exception that the reciprocal case is not true for 12 values of m, which will be presented as follows.

Lemma 3.1. *Let $m \in \mathbb{N}$ and suppose that m admits a t-squared partition. Then we can find $a, b \in \mathbb{N}$ such that $m = b^2 + 2a$ with*

$$a \equiv b \pmod 2 \quad \textit{and} \quad b \leq a \leq b^2 \leq ta.$$

Proof. The fact that $m = b^2 + 2a$ follows from (3.1), and since $c_j^2 \equiv c_j \pmod 2$, for $j = 1, 2, \ldots, t$, we have that $a \equiv b \pmod 2$. Now we focus our attention in proving the inequalities. It is easy to see that a positive integer m admits a *t-squared partition* if m can be written as $m = b^2 + 2a$, and there is a solution for the system

$$\begin{cases} b = x_1 + \cdots + x_t, \\ a = x_1^2 + \cdots + x_t^2, \end{cases} \tag{3.2}$$

with $x_1, \ldots, x_t \in \mathbb{N}$. Since these are all natural numbers it follows easily that $b^2 \geq a \geq b$. The last inequality follows from the Cauchy-Schwarz inequality since

$$b^2 = (\sum_{i=1}^{t} x_i)^2 = (\sum_{i=1}^{t} x_i \cdot 1)^2 \leq (\sum_{i=1}^{t} x_i^2)(\sum_{i=1}^{t} 1^2) = ta.$$

□

Corollary 3.2. *Let $m \in \mathbb{N}$. The integer m admits a t-squared partition only if $m \equiv 0$ or $3 \pmod 4$.*

Proof. It follows from Lemma 3.1 that $m = b^2 + 2a$, with $a \equiv b \pmod 2$. Now it follows from this congruence condition that $m \not\equiv 1$ or $2 \pmod 4$. □

For some special values of m, and also for small values of t is easy to obtain t-squared partitions, as can be seen in the next two results.

Lemma 3.3. *Let m be a positive integer. If $m+1=d^2$, for some $d\in\mathbb{N}$, then m admits a $(d-1)$-squared partition.*

Proof. Let us write $m=d^2-1=(d-1)^2+2(d-1)$. Now take $x_1=\cdots=x_{d-1}=1$ as a solution for the system (3.2), with $t=d-1$ and $a=b=d-1$. □

Lemma 3.4. *Let m be a positive integer written as $m=b^2+2a$. Then*

(a) m admits a 1-squared partition if, and only if, $a=b^2$.

(b) m admits a 2-squared partition if, and only if, $2a-b^2$ is a square smaller than b^2.

Proof. The case (a) is immediate, for the only possibility is to write $m=b^2+2b^2$. Let us proceed to the other case, considering the system (3.2) with $t=2$. Observe that $2a-b^2=2(x_1^2+x_2^2)-(x_1+x_2)^2=(x_1-x_2)^2$. Thus if m admits a 2-*squared partition*, then $2a-b^2=(x_1-x_2)^2$. Since $x_1,x_2\in\mathbb{N}$, we have that $|x_1-x_2|<x_1+x_2=b$. Conversely, consider $2a-b^2=d^2<b^2$ and take $x_1=(b+d)/2$ and $x_2=(b-d)/2$. Since $b\equiv d \pmod 2$ and $b>d$, we have that x_1 and x_2 are positive integers. □

Next we present some combinatorial lemmas that will be helpful for our study of numbers m admitting t-squared partitions.

Lemma 3.5. *Let $c_1,c_2,\dots,c_s\in\mathbb{N}$, with $s\geq 2$, and assume $c_1\geq\cdots\geq c_s$. Then*

$$c_1^2+c_2^2+\cdots c_s^2\leq\left((\sum_{i=1}^{s}c_i)-1\right)^2+1.$$

Proof. The proof is done by induction on s. Let $s=2$, then

$$(c_1+c_2-1)^2+1=(c_1+c_2)^2-2(c_1+c_2)+2\geq c_1^2+c_2^2,$$

since $c_1,c_2\in\mathbb{N}$. Now, let $b=c_1+c_2+\cdots+c_s$. By the induction hypothesis, we have

$$\begin{aligned} c_1^2+\cdots+c_{s-1}^2+c_s^2 &\leq ((b-c_s)-1)^2+1+c_s^2\leq \\ &\leq (b-1)^2+1-2c_s((b-1)-c_s)\leq (b-1)^2+1, \end{aligned}$$

since $b>c_s$. □

Lemma 3.6. *Let $c_1,c_2,\dots,c_s\in\mathbb{N}$, with $s\geq 2$, and assume that they are not all equal. Then*

$$2\sum_{1=i<j}^{s}c_ic_j+(s-1)\leq(s-1)\sum_{i=1}^{s}c_i^2. \tag{3.3}$$

Proof. The proof is done by induction on s. The case $s = 2$ follows from $(c_1 - c_2)^2 \geq 1$. Let us assume that there is only one c_j different from the others, say $c_1 = \cdots = c_{s-1} \neq c_s$. In this case

$$\sum_{1=i<j}^{s} c_i c_j = (\sum_{i=1}^{s-2} i)c_1^2 + (s-1)c_1 c_s,$$

hence the LHS of (3.3) is equal to $(s-1)\{(s-2)c_1^2 + 2c_1 c_s + 1\}$ and the RHS of (3.3) is equal to $(s-1)\{(s-1)c_1^2 + c_s^2\}$. Now it is simple to see that the inequality in (3.3) holds since $(c_1 - c_s)^2 \geq 1$.

Let us assume $c_1 \geq \cdots \geq c_s$ and write $c_j = c_s + \delta_j$, for $j = 1, \ldots, s-1$. Hence we have

$$2\sum_{1=i<j}^{s} c_i c_j = s(s-1)c_s^2 + 2(s-1)c_s(\sum_{j=1}^{s-1} \delta_j) + 2\sum_{1=i<j}^{s-1} \delta_i \delta_j, \tag{3.4}$$

and

$$(s-1)\sum_{i=1}^{s} c_i^2 = s(s-1)c_s^2 + 2(s-1)c_s(\sum_{j=1}^{s-1} \delta_j) + (s-1)\sum_{i=1}^{s-1} \delta_i^2. \tag{3.5}$$

Since the δ_j's are not all equal (for there are at least two distinct c_j's), the result follows from the induction hypothesis, since

$$2\sum_{1=i<j}^{s-1} \delta_i \delta_j + (s-2) \leq (s-2)\sum_{i=1}^{s-1} \delta_i^2 < (s-1)\sum_{i=1}^{s-1} \delta_i^2,$$

(see (3.4) and (3.5) above), completing the proof. □

Theorem 3.7. *Let $m \in \mathbb{N}$. Then m admits a t-squared partition only if m can be written as $m = b^2 + 2a$, with $a, b \in \mathbb{N}$ and*

$$(i) \quad \left\lceil \sqrt{\frac{m}{3}} \right\rceil \leq \; b \; \leq \lfloor \sqrt{m+1} \rfloor - 1.$$

$$(ii) \quad (\left\lceil \frac{b}{t} \right\rceil)^2 + (t-1)(\left\lfloor \frac{b}{t} \right\rfloor)^2 \; \leq \; a \; \leq (b-1)^2 + 1.$$

Proof. Let $m = b^2 + 2a$, and $c_1, \ldots, c_t \in \mathbb{N}$ be a solution for (3.2). From the inequalities stated in Lemma 3.1 we have

$$b^2 + 2b \leq \; m \; \leq 3b^2,$$

which gives (i), since $b^2+2b=(b+1)^2-1$. For the item (ii), the inequality on the RHS follows directly from Lemma 3.5. Now observe that

$$(\lceil\frac{b}{t}\rceil)^2+(t-1)(\lfloor\frac{b}{t}\rfloor)^2=\begin{cases}\dfrac{b^2}{t}, & \text{if } b\equiv 0 \pmod t\\[2ex] \dfrac{((b-r)+1)^2+(t-1)}{t}, & \text{if } b\equiv r\not\equiv 0\,(\text{mod } t).\end{cases}$$

In any case we have, (taking $r=1$)

$$(\lceil\frac{b}{t}\rceil)^2+(t-1)(\lfloor\frac{b}{t}\rfloor)^2\le\frac{b^2+(t-1)}{t}.$$

By Lemma 3.6, we have

$$b^2+(t-1)\le t\sum_{i=1}^{t}c_i^2=\ ta,$$

concluding the proof. □

Next, we present an elementary lemma that will be helpful in proving the main theorem of this section.

Lemma 3.8. *Let $m\in\mathbb{N}$, $m\ge 5$ and let*

$$c(m)=\left\lceil\sqrt{\frac{3m-10}{5}}\,\right\rceil \quad \textit{and} \quad d(m)=\left\lfloor\sqrt{\frac{7m}{9}}\right\rfloor. \tag{3.6}$$

If $m\ge 290$ then $d(m)\ge c(m)+1$.

Proof. Observe that

$$H(m)=\sqrt{\frac{7m}{9}}-\sqrt{\frac{3m-10}{5}}\ \ >(\sqrt{\frac{7}{9}}-\sqrt{\frac{3}{5}})\sqrt{m}>\frac{\sqrt{m}}{10},$$

hence $H(m)$ is an increasing function. Since $H(350)>2$, consequently we have $d(m)\ge c(m)+1$, for $m\ge 350$. For the other values of m in the interval $[290\,,\,349]$, a computer search verified that $d(m)-c(m)\ge 1$, in all of these cases. □

Our goal is to prove that any $m\in\mathbb{N}$, $m\equiv 0$ or $3 \pmod 4$, admits a t-squared partition, provided m is not one of the 12 exceptional values. For this purpose we need the following theorem proved in Pall[10].

Theorem 3.9 (Theorem 4, [10])**.** *Let $a, b \in \mathbb{N}$, and assume that $a \equiv b \pmod 2$ and $7a \geq b^2 \geq 3a-5$. Then the system (3.2), with $t = 7$, has a solution $c_1, \ldots, c_7 \in \mathbb{N} \cup \{0\}$.*

Theorem 3.10. *Let $m \in \mathbb{N}$ such that $m \equiv 0$ or $3 \pmod 4$ then m always admits a t-squared partition, unless*

$$m \in \{4, 7, 11, 16, 20, 23, 31, 40, 44, 55, 68, 95\}.$$

Proof. Let $m \equiv 0 \,\text{or}\, 3 \pmod 4$, and for each value of m, consider the interval $[c(m), d(m)]$, with $c(m), d(m)$ given in (3.6). If this interval contains at least two consecutive integers, then we can choose b within this interval such that $b \equiv m \pmod 2$. Now, it follows from (3.6) that

$$\frac{3}{5}m - 2 \leq \; b^2 \; \leq \frac{7}{9}m, \tag{3.7}$$

Let $a = (m - b^2)/2$, and recall that $m \equiv 0$ or $3 \pmod 4$ and $m \equiv b \pmod 2$. If $m \equiv 0 \pmod 4$, then we also have $b^2 \equiv 0 \pmod 4$, and if $m \equiv 3 \pmod 4$, then b is odd, and $b^2 \equiv 1 \pmod 4$. In any case we have $a \equiv b \pmod 2$.

It follows from (3.7) that a and b satisfy the following inequalites

$$9b^2 \leq 7m \implies 2b^2 \leq 7(m - b^2) \implies b^2 \leq 7a,$$

and

$$3m - 10 < 5b^2 \implies 3(m - b^2) - 10 \leq 2b^2 \implies 3a - 5 \leq b^2.$$

Hence, for this choice of a and b there exist a solution $c_1, \ldots, c_7 \in \mathbb{N} \cup \{0\}$ for the system (3.2) with $t = 7$, according to Theorem **??**. With no loss in generality, let us assume $c_1 \geq \cdots \geq c_7 \geq 0$, and since $b \neq 0$, there must be an t such that $c_1 \geq \cdots \geq c_t \geq 1$ and $c_{t+1} = \cdots = c_7 = 0$. Therefore, m admits a t-squared partition, as desired.

It follows from Lemma 3.8 that the interval $[c(m), d(m)]$ has at least two elements, provided $m \geq 290$. A simple computer search in the interval $[3, 289]$, considering $m \equiv 0 \,\text{or}\, 3 \pmod 4$, reveals that for the values of m listed below, we also have $d(m) \geq c(m) + 1$:

63	83	84	107	108	131	132	135	136	156	159	160	163	164
167	168	187	188	191	192	195	196	199	200	203	204	219	220
223	224	227	228	231	232	235	236	239	240	243	252	255	256
259	260	263	264	267	268	271	272	275	276	279	280	283	284

For each of the remaining 88 integers m in the interval $[3, 289]$ such that $m \equiv 0 \,\text{or}\, 3 \pmod 4$, we proceed as follows (see Theorem 3.7):

1. Determine $R = \lceil \sqrt{\frac{m}{3}} \rceil$ and $S = \lfloor \sqrt{m+1} \rfloor$;
2. Determine the values of b such that $R \leq b \leq S-1$ and $b \equiv m \pmod 2$;
3. Determine a such that $m = b^2 + 2a$;
4. Find a solution for the system below assuming $x_j \leq \sqrt{a}$:

$$\begin{cases} b = x_1 + \cdots + x_t \\ a = x_1^2 + \cdots + x_t^2. \end{cases}$$

The values of m in this interval are small, and it is relatively simple to determine which of these values admit a t-squared partition. Below, we present a short list with the twelve largest values of m in this interval and their respective t-squared partitions:

$$\begin{aligned}
288 &= (1+1+\cdots+1)^2 + 2\times(1^2+1^2+\cdots+1^2) = 16^2 + 2\times 16\\
287 &= (7+2+2+1+1)^2 + 2\times(7^2+2^2+2^2+1^2+1^2) = 13^3 + 2\times 59\\
251 &= (7+4)^2 + 2\times(7^2+4^2) = 11^2 + 2\times 65\\
248 &= (5+5+1+1)^2 + 2\times(5^2+5^2+1^2+1^2) = 12^2 + 2\times 52\\
247 &= (4+4+2+1+1+1)^2 + 2\times(4^2+4^2+2^2+1^2+1^2+1^2) = 13^2 + 2\times 39\\
244 &= (6+3+2+1)^2 + 2\times(6^2+3^2+2^2+1^2) = 12^2 + 2\times 50\\
216 &= (7+3)^2 + 2\times(7^2+3^2) = 10^2 + 2\times 58\\
215 &= (6+3+1+1)^2 + 2\times(6^2+3^2+1^2+1^2) = 11^2 + 2\times 47\\
212 &= (4+3+2+2+1)^2 + 2\times(4^2+3^2+2^2+2^2+1^2) = 12^2 + 2\times 34\\
211 &= (6+2+2+1)^2 + 2\times(6^2+2^2+2^2+1^2) = 11^1 + 2\times 45\\
208 &= (4+3+2+1+1+1)^2 + 2\times(4^2+3^2+2^2+1^2+1^2+1^2) = 12^2 + 2\times 32\\
207 &= (6+2+1+1+1)^2 + 2\times(6^2+2^2+1^1+1^2+1^2) = 11^2 + 2\times 43
\end{aligned}$$

Note that the remaining 76 values of m, which are smaller than 207 and not included among the 56 values listed above, fall within the interval [3,184], and for these values we have found t-squared partitions, unless m is in the set

$$\{4, 7, 11, 16, 20, 23, 31, 40, 44, 55, 68, 95\}.$$

□

Definition 3.11. Let $m \in \mathbb{N}$ and define $f(m)$, the frequency of m, as the number of times m can be represented by a t-squared partition.

Corollary 3.12. *Let $m \in \mathbb{N}$, such that $m \equiv 0$ or $3 \pmod 4$. Then $f(m)$ is equal to the number of non-negative solutions $(c_1, c_2, \ldots, c_b)$, assuming $c_1 \geq c_2 \geq \cdots \geq c_b \geq 0$, of systems of the type*

$$\begin{cases} b = x_1 + \cdots + x_b \\ a = x_1^2 + \cdots + x_b^2, \end{cases} \tag{3.8}$$

for any pair a, b such that $a \equiv b \pmod 2$ and $m = b^2 + 2a$. Moreover, $f(m) \geq 1$ unless $m \in \{4, 7, 11, 16, 20, 23, 31, 40, 44, 55, 68, 95\}$.

Proof. Given a non-negative solution $(c_1, c_2, \ldots, c_b)$ with $c_1 \geq c_2 \geq \cdots \geq c_b \geq 0$, we may assume that for some $t \geq 1$ we have $c_t \neq 0$ and $c_{t+1} = \cdots = c_b = 0$. This shows that m admits a t-squared partition and for any distinct non-negative solution $(c_1, c_2, \ldots, c_b)$ of (3.8), assuming $c_1 \geq c_2 \geq \cdots \geq c_b \geq 0$, we have a distinct t-squared partition of m. The final statement is a direct consequence of Theorem 3.10. □

Example 3.13. *It is not a simple task to determine the frequency of a number, for it involves calculating the number of positive solutions of the system (3.8). But for small values of m it can be easily done, for example, a simple computation shows that $f(107) = 2$ and $f(144) = 4$. Below we have a list of the distinct t-squared partitions of 107 and 144.*

$$107 = (5+2)^2 \;+\; 2 \times (5^2 + 2^2)$$
$$107 = (2+2+1+1+1+1+1)^2 \;+\; 2 \times (2^2 + 2^2 + 1^2 + 1^2 + 1^2 + 1^2 + 1^2)$$

$$144 = (6+2)^2 \;+\; 2 \times (6^2 + 2^2)$$
$$144 = (3+2+2+2+1)^2 \;+\; 2 \times (3^2 + 2^2 + 2^2 + 2^2 + 1^2)$$
$$144 = (3+3+1+1+1+1)^2 \;+\; 2 \times (3^2 + 3^2 + 1^2 + 1^2 + 1^2 + 1^2)$$
$$144 = (4+1+1+1+1+1+1)^2 \;+\; 2 \times (4^2 + 1^2 + 1^2 + 1^2 + 1^2 + 1^2 + 1^2)$$

For the interested reader we recommend the papers of Kloosterman[7] and Pall[10] where conditions for the existence of integer solutions and formulas for the number of integer solutions for the system (3.8) are presented.

4 The Special Set $\mathcal{U}_n(m, \beta, \delta)$

Let $m \in \mathbb{N}$, $m \equiv 0$ or $3 \pmod 4$ and $m \notin \{4, 7, 11, 16, 20, 23, 31, 40, 44, 55, 68, 95\}$, and let us define $\mathcal{A}(m)$ as the set of all non-negative solutions $\vec{x}$ of the system (3.8) such that

$$\vec{x} = (c_1, c_2, \ldots, c_b), \text{ and } c_1 \geq c_2 \geq \cdots \geq c_b \geq 0,$$

for any pair a, b such that $a \equiv b \pmod 2$ and $m = b^2 + 2a$. According to Corollary 3.12, we have $|\mathcal{A}(m)| = f(m) \geq 1$. For $\vec{x} \in \mathcal{A}(m)$, define

- $w(\vec{x})$ as the number of nonzero coordinates of $\vec{x}$;
- $\mu(\vec{x})$ as the sum of the coordinates of $\vec{x}$;
- $\gamma(\vec{x})$ as the biggest coordinate of $\vec{x}$.

Hence if $\vec{x} = (c_1, c_2, \ldots, c_b) \in \mathcal{A}(m)$ and $w = w(\vec{x})$ then

$$\begin{aligned} &c_1 \geq c_2 \geq \cdots \geq c_w \geq 1, \text{ and } c_{w+1} = \cdots = c_b = 0, \\ &\mu(\vec{x}) = \sum_{i=1}^{b} c_i = c_1 + \cdots + c_w \text{ and } \gamma(\vec{x}) = c_1. \end{aligned} \tag{4.1}$$

From this point onwards, we will represent the vector $\vec{x} \in \mathcal{A}(m)$ as $\vec{x} = (c_1, \ldots, c_{w(\vec{x})})$.

For fixed $n \in \mathbb{N}$, and $\beta, \delta \in \mathbb{N} \cup \{0\}$, with $n > \beta$, we define $\mathcal{U}_n(m, \beta, \delta)$ as the subset of $\mathcal{A}(m)$ such that, if $\vec{x} \in \mathcal{U}_n(m, \beta, \delta)$ then

$$\mu(\vec{x}) + \gamma(\vec{x}) + \delta \leq n, \quad c_{w(\vec{x})} \geq \beta, \text{ and } c_j \geq c_{j+1} + \delta, \text{ for } 1 \leq j < w(\vec{x}). \tag{4.2}$$

Lemma 4.1. *Let $n \in \mathbb{N}$, and $\beta, \delta \in \mathbb{N} \cup \{0\}$, with $n > \beta$. If $m \geq n^2$ then*

$$\mathcal{U}_n(m, \beta, \delta) = \emptyset$$

.

Proof. Suppose $m \geq n^2$ and let $\vec{x} \in \mathcal{U}_n(m, \beta, \delta)$. Writing $\vec{x} = (c_1, c_2, \ldots, c_w)$, we have

$$m = (c_1 + c_2 + \cdots + c_w)^2 + 2(c_1^2 + c_2^2 + \cdots + c_w^2). \tag{4.3}$$

Since $\vec{x} \in \mathcal{U}_n(m, \beta, \delta)$, we must have $(c_1 + c_2 + \cdots + c_w) + c_1 + \delta \leq n$, (see (4.2)) and then

$$(c_1 + c_2 + \cdots + c_w)^2 + 2(c_1 + \delta)(c_1 + c_2 + \cdots + c_w) + (c_1 + \delta)^2 \leq n^2.$$

Since

$$(c_1 + c_2 + \cdots + c_w)c_1 \geq (c_1^2 + c_2^2 + \cdots + c_w^2)$$

we must have (see (4.3)) $m < n^2$, a contradiction. Therefore the set $\mathcal{U}_n(m, \beta, \delta)$ must be empty. □

Corollary 4.2. *Let $n \in \mathbb{N}$, and $\beta, \delta \in \mathbb{N} \cup \{0\}$, with $n > \beta$. The set $\mathcal{U}_n(m, \beta, \delta)$ is empty if either*

$$m \equiv 1 \text{ or } 2 \pmod 4, \quad m \in \{4, 7, 11, 16, 20, 23, 31, 40, 44, 55, 68, 95\}, \quad \text{or } m \geq n^2.$$

Proof. These results follow directly from Corollaries 3.2 and 3.12, and Lemma 4.1. □

Lemma 4.3. *Let $n \in \mathbb{N}$, and $\beta, \delta \in \mathbb{N} \cup \{0\}$, with $n > \beta$. If $m_1, m_2 \in \mathbb{N}$, with $m_1 \neq m_2$ then $\mathcal{U}_n(m_1, \beta, \delta) \cap \mathcal{U}_n(m_2, \beta, \delta) = \emptyset$.*

Proof. We may assume that neither m_1 nor m_2 satisfies the conditions stated in Corollary 4.2. Suppose that $m_1 = b_1^2 + 2a_1$, $m_2 = b_2^2 + 2a_2$ and $\vec{x} \in \mathcal{U}_n(m_1, \beta, \delta) \cap \mathcal{U}_n(m_2, \beta, \delta)$. But this implies that $\vec{x} = (c_1, \ldots, c_w)$ and

$$\begin{cases} b_1 = & c_1 + \cdots + c_w & = b_2 \\ a_1 = & c_1^2 + \cdots + c_w^2 & = a_2, \end{cases}$$

which is impossivel since $m_1 \neq m_2$. □

The next theorem establishes an 1-1 correspondence between subsets of $\mathcal{A}(m)$ and subsets of $\mathbb{M}(n, \beta, \delta)$ (see Section 2 above).

Theorem 4.4. *Let $n \in \mathbb{N}$ and $\beta, \delta \in \mathbb{N} \cup \{0\}$, with $n > \beta$. There exits an 1-1 correspondence between vectors $\vec{x}$ in $\bigcup_{m=1}^{n^2-1} \mathcal{U}_n(m, \beta, \delta)$ and two-line matrices M in $\bigcup_{j=1}^{n} \mathbb{M}_0(j, \beta, \delta)$.*

Proof. Let $m \in (1,\, n^2 - 1)$ such that $\mathcal{U}_n(m, \beta, \delta) \neq \emptyset$ (see Corollary 4.2). According to (4.1) and (4.2), for any $\vec{x} \in \mathcal{U}_n(m, \beta, \delta)$, written as $\vec{x} = (c_1, c_2, \ldots, c_w)$ we have

$$c_w = \beta + d_{w+1}, \;\; c_{w-1} = c_w + \delta + d_w, \; \ldots \;, \; c_1 = c_2 + \delta + d_2, \tag{4.4}$$

with $d_2, d_3, \ldots, d_w \in \mathbb{N} \cup \{0\}$. Let us denote $c_{w+1} = \beta$ and associate to the vector $\vec{x}$ the $2 \times (w+1)$ matrix (see (4.4))

$$M(\vec{x}) = \begin{pmatrix} (c_1 + \delta) & (c_2 + \delta) & \cdots & (c_w + \delta) & c_{w+1} \\ 0 & d_2 & \cdots & d_w & d_{w+1} \end{pmatrix}.$$

Hence we have (see (4.4))

$$c_{w+1} = \beta, \;\; c_w + \delta = c_{w+1} + d_{w+1} + \delta, \;\; \text{and} \;\; (c_j + \delta) = (c_{j+1} + \delta) + d_{j+1} + \delta,$$

for $j = 1, 2, \ldots, w-1$. Since (see (4.4))

$$d_2 + \cdots + d_{w+1} = c_1 - (w-1)\delta - \beta,$$

we have (see (4.2))

$$\begin{aligned} \ell(M(\vec{x})) &= \textstyle\sum_{i=1}^{w} (c_i + \delta) + \beta + \sum_{j=2}^{w+1} d_j \\ &= 2c_1 + c_2 + \cdots + c_w + \delta \;=\; \gamma(\vec{x}) + \mu(\vec{x}) + \delta \;\; \leq n. \end{aligned}$$

Hence (see (2.5))

$$M(\vec{x}) \in \mathbb{M}_0(\ell_x, \beta, \delta) \subset \bigcup_{j=1}^{n} \mathbb{M}_0(j, \beta, \delta).$$

Now take a matrix $M \in \mathbb{M}_0(r, \beta, \delta)$, with $r \leq n$,

$$M = \begin{pmatrix} a_1 & a_2 & \cdots & a_s & a_{s+1} \\ 0 & b_2 & \cdots & b_s & b_{s+1} \end{pmatrix}.$$

According to (2.2) we have

$$a_{s+1} = \beta, \quad \text{and} \quad a_j = a_{j+1} + b_{j+1} + \delta \ \geq \ a_{j+1} + \ \delta \ > \ \delta, \quad \text{for} \quad j = 1, 2, \ldots, s. \tag{4.5}$$

Let us define

$$\vec{x}_M = (c_1, \ldots, c_s) \ = \ ((a_1 - \delta), \ldots, (a_s - \delta)),$$

hence, $c_1 \geq c_2 \geq \cdots \geq c_s$ (see (4.5)) and, for $j = 1, 2, \ldots, s-1$,

$$\begin{array}{rl} c_j = & a_j - \delta = a_{j+1} + d_{j+1} = (a_{j+1} - \delta) + d_{j+1} + \delta \ \ \geq \ c_{j+1} + \delta, \quad \text{and} \\ c_s = & a_s - \delta = c_{s+1} + d_{w+1} = \beta + d_{w+1} \ \ \geq \ \beta. \end{array}$$

It follows from Lemma 2.1(iii) and Definition 2.3 that

$$\begin{array}{rcl} \ell(M) & = & a_1 + (\sum_{j=1}^{s} a_j) - s\delta \\ & = & (a_1 - \delta) + (a_1 - \delta) + (a_2 - \delta) + \cdots + (a_s - \delta) + \delta \\ & = & c_1 + c_1 + \cdots + c_s + \delta \\ & = & \mu(\vec{x}) + \gamma(\vec{x}) + \delta \ = r \ \leq n. \end{array} \tag{4.6}$$

Now define

$$m_M = (c_1 + \cdots + c_s)^2 + 2(c_1^2 + \cdots + c_s^2). \tag{4.7}$$

We want to prove that $\vec{x}_M \in \mathcal{U}(m_M, \beta, \delta)$, therefore, the only thing left to be proved is that $m_M \in (1, n^2 - 1)$ (see (4.2)). Observe that (see (4.6))

$$\begin{array}{rl} n^2 \geq \ell(M)^2 = & (\mu(\vec{x}_M) + \gamma(\vec{x}) + \delta)^2, \\ = & (c_1 + \cdots + c_s)^2 + 2(c_1 + \delta)(c_1 + \cdots + c_s) + (c_1 + \delta)^2 \\ > & (c_1 + \cdots + c_s)^2 + 2c_1(c_1 + \cdots + c_s) \\ \geq & (c_1 + \cdots + c_s)^2 + 2(c_1^2 + \cdots + c_s^2) = m_M, \end{array}$$

which completes the proof. □

Example 4.5. *At this point, we would like to present an example illustrating how these correspondences are applied. We will start with the t-squared partitions of* 107*, and conclude with the corresponding partitions of* 12*. From Example 3.13, we have the following representations of* 107*:*

$$\begin{aligned} 107 &= (5+2)^2 \;\; + \;\; 2\times(5^2+2^2) \\ 107 &= (2+2+1+1+1+1+1)^2 \; + \; 2\times(2^2+2^2+1^2+1^2+1^2+1^2+1^2). \end{aligned}$$

Let $\vec{x}_1 = (5,2)$ *and* $\vec{x}_2 = (2,2,1,1,1,1,1)$ *be vectors of* $\mathcal{A}(107)$*. Since*

$$\mu(\vec{x}_1)+\gamma(\vec{x}_1) = 5+5+2 = 12 \;\; and \;\; \mu(\vec{x}_2)+\gamma(\vec{x}_2) = 2+2+2+1+1+1+1+1 = 11,$$

we have that $\vec{x}_1, \vec{x}_2 \in \mathcal{U}_{12}(107,1,0)$*.*

From the correpondence described in the proof of Theorem 4.4 we obtain the matrices

$$M(\vec{x_1}) = \begin{pmatrix} 5 & 2 & 1 \\ 0 & 3 & 1 \end{pmatrix} \in \mathbb{M}_0(12,1,0),$$

and

$$M(\vec{x_2}) = \begin{pmatrix} 2 & 2 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in \mathbb{M}_0(11,1,0).$$

To this last matrix we apply the correspondence described in the proof of Lemma 2.4 to obtain the matrix

$$M_2 = \begin{pmatrix} 2 & 2 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in \mathbb{M}(12,1,0).$$

Finally, from the correspondence between two-line matrices and partitions (see (2.6)) we have the following partitions of 12*.*

$$\begin{aligned} 12 &= 5+5+2, \\ 12 &= 3+2+2+1+1+1+1+1. \end{aligned}$$

Corollary 4.6. *Under the same hypothesis of Theorem 4.4 we have*

$$\sum_{m=1}^{n^2-1} |\mathcal{U}_n(m,\beta,\delta)| = \sum_{j=1}^{n} |\mathbb{M}_0(j,\beta,\delta)|.$$

Proof. It directly follows from Theorem 4.4, Lemma 4.3, and Remark 2.2 that they collectively establish a one-to-one correspondence between unions of disjoint sets. □

5 Main Theorem

Now we are ready to state and prove our main Theorem presenting new formulas for the number of unrestricted partions of n, partitions of n into distinct parts, and the partitions of n arising from the two classic Rogers-Ramanujan Identities.

Theorem 5.1. *Let n be a natural number. Then*

(a) The number of unrestricted partitions of n is equal to $\sum_{m=1}^{n^2-1} |\mathcal{U}_n(m,1,0)| + 1$.

(b) The number of partitions of n into distinct parts is equal to $\sum_{m=1}^{n^2-1} |\mathcal{U}_n(m,1,1)| + 1$.

(c) The number of partitions of n where the difference between two parts is at least two is equal to $\sum_{m=1}^{n^2-1} |\mathcal{U}_n(m,2,1)| + 1$.

(d) The number of partitions of n where the difference between two parts is at least two and each part is greater than one is equal to $\sum_{m=1}^{n^2-1} |\mathcal{U}_n(m,2,2)| + 1$.

Proof. By the definition of $\mathbb{M}(n,\beta,\delta)$, we have that the number of unrestricted partitions of n is equal to $|\mathbb{M}(n,1,0)|$, the number of partitions of n into distinct parts is equal to $|\mathbb{M}(n,1,1)|$, the number of partitions of n where the difference between two parts is at least two is equal to $|\mathbb{M}(n,2,1)|$, and the number of partitions of n where the difference between two parts is at least two and each part is greater than one is equal to $|\mathbb{M}(n,2,2)|$. Now the conclusion follows from Lemma 2.4 and Corollary 4.6, since

$$|\mathbb{M}(n,\beta,\delta)| = \sum_{j=1}^{n} |\mathbb{M}_0(j,\beta,\delta)| + 1 = \sum_{m=1}^{n^2-1} |\mathcal{U}_n(m,\beta,\delta)| + 1.$$

□

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