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A NOTE ON STIRLING NUMBERS OF THE FIRST KIND AND CYCLE TYPES OF A PERMUTATION

Gabriel de Freitas Pinheiro
Universidade Estadual de Campinas, Brazil.
freitasgabriel688@gmail.com
Irene Magalhães Craveiro
Universidade Federal da Grande Dourados.
irenecraveiro@ufgd.edu.br

Abstract

This paper proposes to establish a relationship between the Stirling numbers of the first kind and the cycle types of S_n, exhibiting the feasibility of a procedure to generate Stirling numbers of the first kind and proving some identities by the combination of these two concepts. This is possible due these numbers' strong algebraic appeal, given that we can define them as the number of permutations of S_n that decompose into exactly k disjoint cycles. There is a bijective relationship between the cycle types of S_n and the partitions of a positive integer n, thus given a partition of n, we know how many permutations of S_n exist that are of a given cycle type. Given that all permutation of S_n can be decomposed into product of cycles, so we know how the number of permutations with a certain cycle type by looking at the integer partitions of n. Thus, Stirling numbers of the first kind can be easily determined. Throughout the article, we will explore some identities concerning Stirling numbers of the first kind and the binomial coefficient, as well as presenting the concepts of partitioning positive integers and cycle types of S_n.

1 Introduction

It is possible to combinatorically think of a permutation of n objects as ordered lists. For instance, if we want to sort the numbers $1, 2, 3$ without any restriction or a specified order, we can do it as follows: $(1, 2, 3), (1, 3, 2), (2, 1, 3), (2, 3, 1), (3, 1, 2), (3, 2, 1)$. That is, we have 6 ways to sort them. If we denote P_n as the number of permutations of n elements, then we have that $P_n = n!$ [5]. From group theory's standpoint, a permutation is a function defined in a set X which takes values also in X. Throughout this work we employ this notion of a permutation, as formalized in Definition 1.1 below.

Definition 1.1. Let X be a non-empty set. A permutation of X is a bijective function $\sigma : X \to X$. We denote by $S(X)$ the set of all permutations of X.

Considering the composition operation, we have that $(S(X), \circ)$ is a group, called a permutation group. In our work, we are interested in the case where the set X is finite. Without loss of generality, we consider that $X = \{1, 2, \cdots, n\} = [n]$. Thus, when X has n elements, we denote its group of permutations by S_n and we call it the symmetric group of degree n. Trivially, $S_n = n!$.

Given $\sigma \in S_n$ and $j \in [n]$, we denote by $\sigma(j)$ the value of the bijective function σ under j. Whilst equipped with this notion, we can represent a permutation of S_n as a two-line diagram shown below:

$$\begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma(1) & \sigma(2) & \cdots & \sigma(n) \end{pmatrix}$$

For example, the identity permutation of S_3 can be represented as $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$ and tells us that $\sigma(1) = 1, \sigma(2) = 2, \sigma(3) = 3$.

Another notation for this permutation would be (1)(2)(3). We call this a cycle notation. Thus, the identity can be represented by the product of three cycles, each with length 1.

Let's see another example, consider the permutation of S_6: $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 5 & 4 & 3 & 1 & 6 \end{pmatrix}$. Notice that the 1 is sent to 2, the 2 is sent to 5, and the 5 is sent back to the 1, as we close the cycle (1 2 5). In the same way, the 3 is sent to 4, and the 4 is sent back to the 3, closing the cycle (3 4). Finally, the 6 is fixed in its original position. Therefore, we have $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 5 & 4 & 3 & 1 & 6 \end{pmatrix} = (1\ 2\ 5)(3\ 4)(6)$.

Therefore, every permutation of S_n can be represented as the product of disjoint cycles, where disjunct means that, given two cycles, their intersection is the empty set.

This idea of cycles of permutations generates a family of numbers known as Stirling numbers of the first kind, which can be defined as the number of permutations of S_n that decompose into exactly k cycles. We give them a proper definition and some necessary background in section 3.

With the concepts defined so far, we explore the cycle types of S_n in the next section, as well as showing that there is a direct relationship between the partitions of the positive integer n with the cycle types of S_n.

In Section 3 we present some identities for certain classes of Stirling numbers of the first kind via cycle types. Finally, in Section 4, we will show an identity for these numbers by means of cycle types.

2 Cycle types of a permutation and the integer partitions of n

As previously mentioned, the set of permutations of n objects is a group when equipped with the composition operation. In [1], the permutations of n objects, S_n, are represented through the product of disjoint cycles and this representation is unique, except by the order of the elements of the cycles. A permutation $\sigma \in S_n$ may have cycles of the same length represented as the product of cycles. Accordingly, we define the cycle type of a permutation.

Definition 2.1. Let $\sigma \in S_n$. The cycle type of σ is defined by $T(\sigma) = x_{l_1}^{k_{l_1}} \dots x_{l_s}^{k_{l_s}}$, where k_{l_j} is the number of cycles of length l_j that appear in the decomposition of σ, $j = 1, \cdots, s$.

Given $\sigma \in S_n$, we can rewrite a cycle type of σ, $T(\sigma) = x_{l_1}^{k_{l_1}} \dots x_{l_s}^{k_{l_s}}$, as $T(\sigma) = x_1^{k_1} \cdots x_n^{k_n}$, where k_j is the number of cycles of length j, some k_j may be null, for $j = 1, 2, \cdots, n$.

Example 2.2. Consider the permutation $\sigma_1 = (4)(2\ 3)(5\ 1)(6\ 8\ 7) \in S_8$, and note that it has one cycle of length 1, two cycles of length 2, and one cycle of length 3, thus the cycle type of σ is given by $T(\sigma_1) = x_1 x_2^2 x_3$.

Example 2.3. Let $\sigma_2 = (7)(2\ 3)(5\ 1)(6\ 8\ 2)$ be a permutation of S_8. The cycle type of σ_2 is given by $T(\sigma_2) = x_1 x_2^2 x_3$.

In Examples 2.2 and 2.3, we have two distinct permutations of S_8 with the same cycle type. From this, it can be seen that the cycle type of a permutation is not unique. Also, note that both permutations σ_1 and σ_2 can be associated with the following integer partition of 8, $1 + 2 + 2 + 3$. Therefore, we aim to establish a connection between the cycle types of S_n and the integer partitions of the positive integer n.

Definition 2.4. Let n be a positive integer. A sequence of integers $(\lambda_1, \lambda_2, \dots, \lambda_s)$, with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_s$, is an integer partition of n if, and only if, $n = \lambda_1 + \lambda_2 + \cdots + \lambda_s$.. The λ_j are called parts of the integer partition, $j = 1, \dots, s$, and s is the length of the partition

Example 2.5. Table 1 shows all the integer partitions of numbers 3, 4, and 5.

We denote by $p(n)$ the number of integer partitions of the positive integer n. Thus, from Table 1 we have that $p(3) = 3$, $p(4) = 5$ and $p(5) = 7$. If $n = 7$, we have $p(7) = 15$, and it is important to notice that the number of integer partitions grow rapidly.

n	3	4	5
Partitions of n	3 2+1 1+1+1	4 3+1 2+2 2+1+1 1+1+1+1	5 4+1 3+2 3+1+1 2+2+1 2+1+1+1 1+1+1+1+1

Table 1: Partitions for $n = 3, 4, 5$

For many years, mathematicians sought to obtain an explicit formula for $p(n)$. Due to the works of S. Ramanujan, G. H. Hardy, and H. Radamacher, we have an asymptotic expression for $p(n)$ given by $\frac{1}{4n\sqrt{3}}e^{\pi\sqrt{\frac{2n}{3}}}$ as $n \to \infty$.

Note that since $\sigma \in S_n$ is a permutation of the elements of $[n]$, then we have n symbols distributed among all the cycles of their equivalent class. Thus, $k_{l_1}l_1 + \cdots + k_{l_s}l_s = n$ and, as $1 \leq l_1 < \cdots < l_j \leq n$, and by the definitions of k_{l_j}, we have an integer partition of $n = k_{l_1}l_1 + \cdots + k_{l_s}l_s$.

Conversely, if we have an integer partition of $n > 1$ with

$$n = k_{l_1}l_1 + \cdots + k_{l_s}l_s, \text{ with } k_{l_j}, l_j \in [n] \text{ and } l_1 < \cdots < l_s$$

with k_{l_j} terms equal to l_j, for $j = 1, \cdots, s$, then it is possible to find a permutation $\sigma \in S_n$ with cycle type $T(\sigma) = x_{l_1}^{k_{l_1}} \cdots x_{l_s}^{k_{l_s}}$.

Theorem 2.6. *Let n be a positive integer such that $n = k_{l_1}l_1 + k_{l_2}l_2 + \cdots + k_{l_s}l_s$. Then, there exist $\frac{n!}{l_1^{k_{l_1}}k_{l_1}!l_2^{k_{l_2}}k_{l_2}! \cdots l_s^{k_{l_s}}k_{l_s}!}$ permutations $\sigma \in S_n$ with cycle type $T(\sigma) = x_{l_1}^{k_{l_1}} \cdots x_{l_s}^{k_{l_s}}$. In other words, given a cycle type $x_1^{k_1} \cdots x_n^{k_n}$, we have*

$$h(k_1, \cdots, k_n) = \frac{n!}{1^{k_1}k_1! \cdots n^{k_n}k_n!}$$

permutations in S_n with this cycle type.

Proof. Consider any permutation of S_n given by $\sigma = (a_1\ a_2\ \cdots\ a_r)$ with $r \leq n$ and a cyclic structure given by $x_1^{k_1}x_2^{k_2} \cdots x_n^{k_n}$.. As previously mentioned, $1k_1+2k_2+\cdots+nk_n = n$, and some k_j can be equal to zero, for $j = 1, \cdots, n$.

Since σ is a permutation of n objects, then the number of ways to rearrange the elements of σ is $n!$. However, the k_i cycles of length i contribute to a rearrangement which can be done by permuting the cycles of the same length. Thus, by the Multiplicative Principle, we have that these cycles contribute with $k_1!k_2!\cdots k_n!$ rearrangements.

Now, a cycle of length r similar to σ can be represented by r different ways, by simply permuting the a_i's of the cycle in the following ways:

$$(a_1\ a_2\ \cdots\ a_r) = (a_r\ a_1\ \cdots\ a_{r-1}) = \cdots = (a_{r-1}\ a_r\ \cdots\ a_1).$$

Therefore, these cycles contribute with $1^{k_1}2^{k_2}\cdots n^{k_n}$ rearrangements, as there are k_i cycles of length i for each i. It follows that

$$h = \frac{n!}{1^{k_1}k_1!\cdots n^{k_n}k_n!}$$

where h is the number of permutations of S_n with cycle type $x_1^{k_1}x_2^{k_2}\cdots x_n^{k_n}$.

□

Given the facts stated so far, we can establish the relationship between the permutations cycle types of S_n and the integer partitions of the positive integer n.

Corollary 2.7. There is a bijective correspondence between the cycle types of S_n and the integer partitions of n. That is, there are $p(n)$ different cycle types of the permutations of S_n.

Example 2.8. Table 2 shows the integer partitions of 4, as well as the permutations of S_4 and the cycle types associated with each group of permutations with the same number of cycles.

Example 2.9. By Theorem 2.6 we know that the number of permutations with cycle type x_4 is $\dfrac{4!}{4^1\cdot 1!} = 3! = 6$. That is, we have 6 permutations of S_4 with cycle type x_4, as shown in Table 2.

Example 2.10. Due to Table 2, we can check that the different cycle types of S_4 are: $x_1^4, x_2^2, x_1^2x_2, x_1x_3, x_4$. In other words, we have $p(4) = 5$ different cycle types, as described by Corollary 2.7.

partitions of 4	permutations of S_4	cycle types
4	$(2341), (2413), (3421), (1342), (4312), (4123)$	x_4
1+3	$(1)(234), (1)(243), (2)(134), (2)(143), (3)(124), (3)(142), (4)(123), (4)(132)$	x_1x_3
2+2	$(12)(34), (13)(24), (14)(23)$	x_2^2
1+1+2	$(1)(2)(34), (1)(3)(24), (1)(4)(23), (2)(3)(14), (2)(4)(13), (3)(4)(12)$	$x_1^2x_2$
1+1+1+1	$(1)(2)(3)(4)$	x_1^4

Table 2: The permutations of S_4 together with the partitions of 4 and their cycle types

3 Stirling numbers of the first kind: definitions and identities

From the point of view of generative functions, the Stirling numbers of the first kind are defined by [3] and [8] as the coefficients of x^k in the expansion of the polynomial $x(x+1)\cdots(x+(n-1))$, with $0 < k \leq n$. That is, they represent a sequence generated by this class of polynomials with fixed degree n, where the coefficients of the powers of x belong to this sequence.

In addition to this definition, we can also interpret Stirling's numbers of the first kind as being the number of ways of n people sit around of k identical circular tables without any table being empty, as defined by [2]. However, we will explore the definition given by [7], where the Stirling numbers of the first kind are equal to the number of permutations of S_n that decompose into exactly k disjoint cycles.

Thus, in this section we will explore Stirling numbers of the first type, as well as present some identities involving these numbers. We will prove these identities using algebraic arguments involving the cycle types of a permutation and the integer partitions of n.

Definition 3.1. Let n, k natural numbers. We define the Stirling numbers of the first kind, denoted by $\left[\begin{array}{c} n \\ k \end{array}\right]$, as the non-negative integers that determine the number of permutations of S_n that decompose into exactly k disjoint cycles. For theoretical reasons, it is agreed upon that, $\left[\begin{array}{c} 0 \\ 0 \end{array}\right] = 1$ and, if $k > 0$ then $\left[\begin{array}{c} 0 \\ k \end{array}\right] = 0$.

Example 3.2. Table 3 shows some values for the Stirling numbers of the first kind.

Table 3: Some Stirling numbers of the first kind

$\left[\begin{array}{c} n \\ k \end{array}\right]$	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = 6$
$k = 1$	1	1	2	6	24	120
$k = 2$	0	1	3	11	50	274
$k = 3$	0	0	1	6	35	225
$k = 4$	0	0	0	1	10	85
$k = 5$	0	0	0	0	1	15
$k = 6$	0	0	0	0	0	1

In Proposition 3.3 we present a result that will be useful in order to prove that the sum of the Stirling numbers of the first kind for a fixed n is $n!$. It follows from a result presented at [4]. An example can also be seen in Table 3, where the sum of the elements of the n-th column is equal to $n!$.

Proposition 3.3. Consider $k_1, k_2, \cdots, k_n \in \{0\} \cup [n]$. Then, we have that

$$\sum_{k_1 + k_2 \cdot 2 + \cdots k_n \cdot n = n} \frac{1}{1^{k_1} k_1! \cdots n^{k_n} k_n!} = 1.$$

Equipped with the concept above, we prove the following result through the definition of cycle types and Corollary 2.7.

Theorem 3.4. *Let the integer* $n \geq 1$. *Then,*

$$\sum_{k=1}^{n} \left[\begin{array}{c} n \\ k \end{array}\right] = n!.$$

Proof. By Corollary 2.7 we have that there are $p(n)$ different permutations cycle types of S_n. Let $\lambda_i = k_1^{(i)} \cdot 1 + k_2^{(i)} \cdot 2 + \cdots + k_n^{(i)} \cdot n$ be the integer partitions of n associated to the $p(n)$ different cycle types. So, by Theorem 2.6 we have that the number of permutations with cycle type $x_1^{k_1^{(i)}} \cdots x_n^{k_n^{(i)}}$ is $\dfrac{n!}{1^{k_1^{(i)}} k_1^{(i)}! \cdots n^{k_n^{(i)}} k_n^{i}!}$.

By definition, we have that $\left[\begin{array}{c} n \\ k \end{array} \right]$ is equal to the number of permutations of S_n that decompose into exactly k disjoint cycles. Thus, for each $k = 1, \cdots, n$, we have that the permutations of S_n with k cycles are also associated with the partitions λ_i. In this way, we have that

$$\begin{aligned} \sum_{k=1}^{n} \left[\begin{array}{c} n \\ k \end{array} \right] &= \sum_{i=1}^{p(n)} \frac{n!}{1^{k_1^{(i)}} k_1^{(i)}! \cdots n^{k_n^{(i)}} k_n^{i}!} \\ &= n! \sum_{i=1}^{p(n)} \frac{1}{1^{k_1^{(i)}} k_1^{(i)}! \cdots n^{k_n^{(i)}} k_n^{i}!} \\ &= n! \end{aligned}$$

The equality between the second and the third lines follows from Proposition 3.3 □

Binomial numbers can be combinatorically interpreted as being the number of ways to choose k objects from a set with n objects. Likewise, in the following propositions we will present some identities involving the Stirling numbers of the first kind and the binomial numbers. It is worth noting that we will prove the propositions with arguments using the integer partitions of n and the cycle types of S_n associated with those integer partitions.

Proposition 3.5. For every $n > 1$, we have that $\left[\begin{array}{c} n \\ n-1 \end{array} \right] = \binom{n}{2}$.

Proof. By definition, we want to find the number of permutations of S_n that decompose as the product of $n-1$ cycles. By Corollary 2.7, we have that, for each integer partition of n, exists a permutation of S_n associated with it. So, we only have to check for integer partitions of n that contain $n-1$ parts. If we consider all the parts that are equal to 1, then we will have an integer partition of $n-1$. So, we need to add 1 to any of the parts, so that we have a partition of n. That is, we will have a part equal to 2 and $n-2$ parts equal to 1. Thus, the cycle type associated with this integer partition is of the form $x_1^{n-2}x_2$. Therefore, by Theorem 2.6, we have that

$$\left[\begin{array}{c} n \\ n-1 \end{array} \right] = \frac{n!}{1^{n-2}(n-2)!2^1 1!} = \frac{n!}{2(n-2)!} = \binom{n}{2}.$$

□

Proposition 3.6. For every $n > 2$, we have that $\left[\begin{matrix} n \\ n-2 \end{matrix}\right] = \frac{3n-1}{4}\binom{n}{3}$.

Proof. In the same way as argued before in the demonstration of the Proposition 3.5, we now need to look at the partitions of n into $n-2$ parts. Considering all the parts equal to 1, we have an integer partition of $n-2$, so we need to add 2 to the integer partition. However, the 2 can be written as 2 and $1+1$, thus the partitions of n with $n-2$ parts are of the form $\underbrace{1+1+\cdots+1}_{n-3}+3$ and $\underbrace{1+1+\cdots+1}_{n-4}+2+2$. The cycle types associated with these partitions are, respectively, $x_1^{n-3}x_3$ and $x_1^{n-4}x_2^2$. Therefore, by Theorem 2.6, we have that

$$\left[\begin{matrix} n \\ n-2 \end{matrix}\right] = \frac{n!}{3(n-3)!} + \frac{1}{4}\frac{n!}{2(n-4!)} = 2\binom{n}{3} + \frac{3(n-3)}{4}\binom{n}{3} = \frac{3n-1}{4}\binom{n}{3}.$$

□

Proposition 3.7. For every $n > 3$, we have that $\left[\begin{matrix} n \\ n-3 \end{matrix}\right] = \binom{n}{2}\binom{n}{4}$.

Proof. The integer partitions of the number 3 are $3, 2+1, 1+1+1$.. Thus, the integer partitions associated with the permutations of S_n that decompose into $n-3$ cycles are $\underbrace{1+1+\cdots+1}_{n-4}+4$, $\underbrace{1+1+\cdots+1}_{n-5}+2+3$ and $\underbrace{1+1+\cdots+1}_{n-6}+2+2+2$. The cycle types are, respectively, $x_1^{n-4}x_4, x_1^{n-5}x_2x_3$ and $x_1^{n-6}x_2^3$. Therefore, by Theorem 2.6, we have that

$$\begin{aligned}\left[\begin{matrix} n \\ n-3 \end{matrix}\right] &= \frac{n!}{4(n-4)!} + \frac{n!}{6(n-5!)} + \frac{n!}{48(n-6)!} \\ &= 6\binom{n}{4} + 4(n-4)\binom{n}{4} + \frac{(n-4)(n-5)}{2}\binom{n}{4} \\ &= \frac{12+8n-32+n^2-9n+20}{2}\binom{n}{4} = \frac{n^2-n}{2}\binom{n}{4} = \binom{n}{2}\binom{n}{4}.\end{aligned}$$

□

Proposition 3.8. For every $n > 4$, we have that $\left[\begin{matrix} n \\ n-4 \end{matrix}\right] = \frac{15n^3-30n^2+5n+2}{48}\binom{n}{5}$.

Proof. On Table 1 we have that the partitions of 4 are: $4, 3+1, 2+2, 2+1+1, 1+1+1+1$. Thus, the cycle types associated with the permutations of S_n that decompose into $n-4$ cycles are of the form $T(\sigma_1) = x_1^{n-5}x_5, T(\sigma_2) = x_1^{n-6}x_2x_4, T(\sigma_3) = x_1^{n-6}x_3^2, T(\sigma_4) = x_1^{n-7}x_2^2x_3$ and $T(\sigma_5) = x_1^{n-8}x_2^4$.Therefore, from Theorem 2.6, we have that

$$\begin{aligned}
\begin{bmatrix} n \\ n-4 \end{bmatrix} &= \frac{n!}{5(n-5)!} + \frac{n!}{8(n-6)!} + \frac{n!}{18(n-6)!} + \frac{n!}{24(n-7)!} + \frac{n!}{384(n-8)!} \\
&= 24\binom{n}{5} + 15(n-5)\binom{n}{5} + \frac{20(n-5)}{3}\binom{n}{5} + 5(n-5)(n-6)\binom{n}{5} + \\
&\quad \frac{5(n-5)(n-6)(n-7)}{16}\binom{n}{5} \\
&= 24\binom{n}{5} + \frac{65(n-5)}{3}\binom{n}{5} + 5(n-5)(n-6)\binom{n}{5} + \\
&\quad \frac{5(n-5)(n-6)(n-7)}{16}\binom{n}{5} \\
&= \frac{1152 + 1040(n-5) + 240(n-5)(n-6) + 15(n-5)(n-6)(n-7)}{48}\binom{n}{5} \\
&= \frac{15n^3 - 30n^2 + 5n + 2}{48}\binom{n}{5}.
\end{aligned}$$

□

In this section we explored some identities involving certain classes of Stirling numbers of the first kind and the binomial coefficient. In the proposition's proofs we could see how integer partitions of n can be used to count these numbers.

It is important to notice that we first looked at the integer partition of $n-k, k = 1, \cdots, 4$, with all parts equal to 1. Thus, to be an integer partition of n, we would need to add k to that partition. However, there are $p(k)$ ways to do so. We looked at the integer partitions of k and then we distributed elements in order to satisfy the definition.

By doing this distribution, we got to know which are the integer partitions of n with exactly $n-k$ parts. Thus, it was enough to simply find the cyclic types associated with these partitions, using the Theorem 2.6 to find the Stirling number of the first type $\begin{bmatrix} n \\ n-k \end{bmatrix}$, for $k = 1, \cdots, 4$.

In the next section we are going to present an identity that identifies the Stirling numbers of the first kind via integer partitions of n and the cycle types of S_n. This idea is similar to other ideas that we have already developed in the course of the article.

4 Explicit formula for the Stirling numbers of the first kind

As we've mentioned before, Stirling numbers of the first kind can be interpreted as the number of permutations of S_n that decompose into exactly k disjoint cycles.

If we take the integer partitions of n with k parts, we also get, by Corollary 2.7, the cycle types associated with these permutations. In other words, we will have the cycle types of all permutations of S_n that decompose to exactly k cycles.

By having these cycle types, we can use Theorem 2.6 to find the number of permutations with certain cycle types. Thus, in the following theorem, we present the main result of this work. Its proof is derived by applying Corollary 2.7 and Theorem 2.6. Consider $p(n,k)$ as the number of integer partitions of n with exactly k parts.

Theorem 4.1. *Let $\pi_i = k_1^{(i)} \cdot 1 + k_2^{(i)} \cdot 2 + \cdots + k_n^{(i)} \cdot n$, for $i = 1, \cdots, p(n,k)$, as integer partitions of n into exactly k parts. Then, for $0 \leq k < n$, we have that*

$$\left[\begin{array}{c} n \\ k \end{array} \right] = \sum_{i=1}^{p(n,k)} \frac{n!}{1^{k_1^{(i)}} k_1^{(i)}! \cdots n^{k_n^{(i)}} k_n^{(i)}!}$$

Example 4.2. In order to find the value of $\left[\begin{array}{c} 5 \\ 2 \end{array} \right]$, we know, from Table 1 that the integer partitions of 5 with exactly 2 parts are: $1+4, 2+3$. Thus, $p(5,2) = 2$. From Theorem 4.1, it follows that

$$\left[\begin{array}{c} 5 \\ 2 \end{array} \right] = \sum_{i=1}^{2} \frac{n!}{1^{k_1^{(i)}} k_1^{(i)}! \cdots n^{k_n^{(i)}} k_n^{(i)}!} = \frac{5!}{1 \cdot 4} + \frac{5!}{2 \cdot 3} = \frac{120}{4} + \frac{120}{6} = 30 + 20 = 50.$$

Consider $p(n,k)$ as the number of integer partitions of n with exactly k parts. If we look at the integer partitions of $n-1$ with $k-1$ parts and add the number 1 in the last part, then we would have an integer partition of n with k parts.

Now, if we take each partition of $n-1$ with k parts and add 1 to some of the parts of the integer partition, then we will also have integer partitions of n into k parts. Note that 1 can be added in $n-1$ ways. Thus, we have the following recurrence for the number of integer partitions of n into exactly k parts:

$$p(n,k) = p(n-1,k-1) + (n-1)p(n-1,k). \tag{4.1}$$

Due to what we have just discussed, it follows from the Corollary 4.3 below a recurrence relationship for the Stirling numbers of the first kind. It is worth noting that the proof for this corollary follows directly from Theorem 4.1 and Equation 4.1. Another proof, using the same arguments as we done for Equation (4.1), can be seen in [6].

Corollary 4.3. Let n, k be positive integers such that $1 < k < n$. Then,

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + (n-1) \begin{bmatrix} n-1 \\ k \end{bmatrix}.$$

5 Closing remarks

As discussed, Stirling numbers of the first kind represent the number of permutations of S_n that decompose into exactly k disjoint cycles. For certain values of k, these numbers are closely linked to the binomial coefficient, which is a powerful combinatorics tool for counting certain groupings of objects.

Broadly speaking, we can think of Stirling numbers of the first kind as a generalization of the notion of a circular permutation, because these numbers count the number of possibilities of distributing n people around k identical round tables, with no table left empty, as well as defines [2].

Based on this definition, we can use concepts from Partition Theory to determine the Stirling numbers of the first kind, and for that we just have to look at the integer partitions of n into at most k parts, and so, by using combinatorics, count the distributions that must be made around the tables.

Thus, we can already see the potential use of Partition Theory to count Stirling numbers of the first kind. However, as we have seen throughout this article, integer partitions of n are closely related to the cycle types of the permutations of S_n, and we know how to count the number of cycle types. In this way, we have established an explicit formula which counts the Stirling numbers of the first kind via integer partitions and cycle types of the permutations of S_n.

Some of our references presented proofs for identities involving the Stirling numbers of the first kind analytically, or by using other means, such as the Vieta's Formula. Others use the recurrence relationship that these numbers have, and, by means of the Principle of Finite Induction, they complete some proofs. However, our intention was to use the algebraic potential that Stirling numbers have and, through Partition Theory and cycle types, establish a way to compute them without having to decompose all the permutations of S_n.

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