

A NOTE ON GENERALIZED NARAYANA SEQUENCE

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Abstract

The purpose of this article is to present a new sequence of numbers using the concept binomial coefficient. This new sequence is a generalization of Narayana's numbers. From this definition we will establish some properties, we will prove some identities and establish connections between this new sequence and the Catalan numbers. Finally, the generalized Narayana central numbers are defined and for some special cases, such as $r = 1, 2, 3$ and 4 , identities involving the generalized Narayana central numbers and Catalan numbers are given and thus we establish a formula for the central Narayana numbers in terms of the Catalan numbers.

Keywords: Generalized Narayana sequence, Catalan Numbers, Generalized Narayana Central Numbers.

1 Introduction

Narayana numbers are closely related to binomial coefficients and according for [2], [5] and [4] this numerical sequence forms a triangular matrix of natural numbers, called the Narayana triangle, in whose rows or columns interesting identities appear.

Furthermore, several counting problems are enumerated through this sequence of numbers, if the reader is curious, see [3] and [4]. In addition, through this sequence it is possible to establish and prove several properties in combinatorics, geometry and algebra (sequence A001263 in [7]).

The Narayana numbers can be defined through a function $N(n, k)$ of two variables, such that n and k which are integers non-negative where the binomial coefficient appears in terms of the parameters n and k . Some authors define $N(0, 0) = 1$, $N(n, 0) = 1$ and $N(n, k) = 0$, if $n < k$.

Now consider positive integer numbers n and k with $1 \leq k \leq n$. The Narayana sequence numbers, in [2], $\{N(n, k)\}_{1 \leq k \leq n}$ is defined by

$$N(n, k) = \frac{1}{n} \binom{n}{k-1} \binom{n}{k}. \quad (1.1)$$

Rewriting (1.1) we obtain

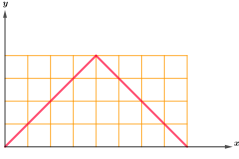
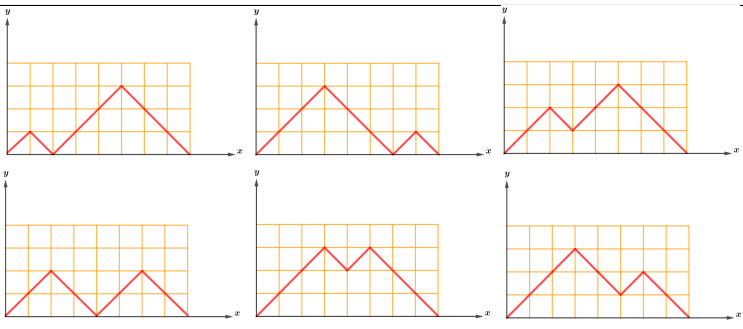
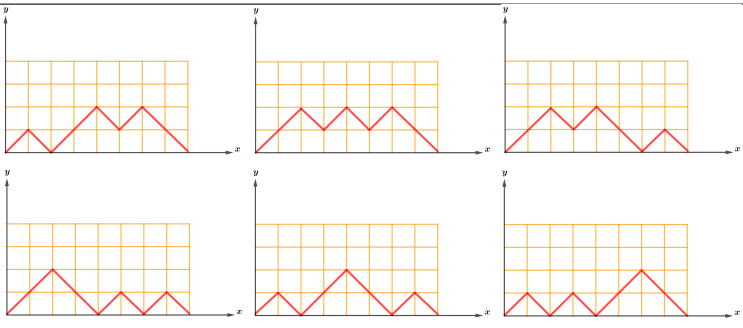
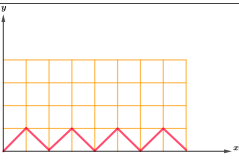
$$N(n, k) = \frac{1}{k} \binom{n-1}{k-1} \binom{n}{k-1}. \quad (1.2)$$

For the development of this article we consider the sequence of Narayana numbers given by (1.2). This numerical sequence forms a triangular matrix of natural numbers, as can be seen in the table below (sequence A001263 in the [7]).

Table 1: Triangle of Narayana Numbers

n \ k	1	2	3	4	5	6
1	1					
2	1	1				
3	1	3	1			
4	1	6	6	1		
5	1	10	20	10	1	
6	1	15	50	50	15	1

To illustrate, we will describe a combinatorial interpretation for Narayana numbers, that is, Narayana numbers $N(n, k)$ count the set of paths of length $2n$ having exactly k peaks. To do this, simply consider the checkerboard mesh $2n \times n$ in the first quadrant of the XOY plane. Leaving $(0, 0)$ we want to reach $(2n, 0)$ where we can make upward or downward movements at the points of the $2n \times n$ grid mesh. Let's illustrate the case $n = 4$ with $1 \leq k \leq 4$.

$1 \leq k \leq 4$	the paths of length $2n$ having exactly k peaks	$N(4, k)$
$k = 1$		$N(4, 1) = 1$
$k = 2$		$N(4, 2) = 6$
$k = 3$		$N(4, 3) = 6$
$k = 4$		$N(4, 4) = 1$

In the literature, Catalan numbers can be defined using the binomial coefficient by

$$C_n = \frac{1}{n+1} \binom{2n}{n},$$

this sequence is formed by the following numbers 1, 1, 2, 5, 14, 42, 132, 429, \dots (sequence A000108 in [7]).

Catalan numbers are closely linked to Narayana numbers through several identities, for example,

$$\sum_{k=0}^n N(n, k) = C_n.$$

In this article, we introduce and analyze a generalized form of the Narayana sequence, defined using binomial coefficients. Our objective is to establish identities for this newly defined sequence and explore its relationship with Catalan numbers.

The paper is organized as follows: Section 2 introduces the generalized Narayana sequence, defined using binomial coefficients, and establishes some properties. In Section 3, we present identities that link Catalan numbers with the generalized Narayana sequence. Additionally, we investigate the generalized central Narayana numbers and establish identities for specific cases. Finally, we conclude with a summary of our findings and suggest directions for future research.

2 The generalized Narayana sequence and some properties

Some generalizations of the Narayana sequence numbers have been proposed, and several general identities have been established (see, for instance, references [1, 2]). In this section, we will define the generalized Narayana sequence and explore some of its properties and relations to the sequence of Narayana numbers defined by (1.2).

The preceding discussion enables us to formulate an expression, involving a non-negative integer r , which provides a generalization for the Narayana numbers. These generalized Narayana numbers will be referred to as such.

Definition 2.1. Given $r \geq 0$, consider the generalized Narayana sequence defined by Expression

$$N_r(n, k) = \frac{1}{n - k + 1} \binom{n + r - 1}{k + r - 1} \binom{n + r - 2}{k + r - 2}, \quad (2.1)$$

for $0 \leq k \leq n$. In the case where $n < k$, we define $N_r(n, k) = 0$. By convention we define the following binomial numbers $\binom{-1}{-1} = \binom{-2}{-2} = 1$.

We will present some properties that follow from the definition of the generalized Narayana number.

Proposition 2.2. Consider the Narayana sequence defined by Expression (1.2). Then,

$$N_0(n + 1, k + 1) = N(n, k), \quad (2.2)$$

for $1 \leq k \leq n$.

Proof. In fact, making $r = 0$ in the explicit formula given in (2.1) we have

$$\begin{aligned}
N_0(n+1, k+1) &= \frac{1}{n+1-k-1+1} \binom{n+1-1}{k+1-1} \binom{n+1-2}{k+1-2} \\
&= \frac{1}{n-k+1} \binom{n}{k} \binom{n-1}{k-1} \\
&= \frac{1}{n-k+1} \frac{n!}{(n-k)!k!} \binom{n-1}{k-1} \\
&= \frac{n!}{k(n-k+1)(n-k)!(k-1)!} \binom{n-1}{k-1} \\
&= \frac{1}{k} \binom{n-1}{k-1} \binom{n}{k-1} \\
&= N(n, k).
\end{aligned}$$

□

Proposition 2.3. Consider the Narayana sequence defined by Expression (1.2). Then,

$$N_1(n, k) = N(n, k), \quad (2.3)$$

for $1 \leq k \leq n$.

Proof. In fact, making $r = 1$ in the explicit formula given in (2.1) we have

$$\begin{aligned}
N_1(n, k) &= \frac{1}{n-k+1} \binom{n}{k} \binom{n-1}{k-1} \\
&= \frac{1}{n-k+1} \cdot \frac{n!}{k!(n-k)!} \binom{n-1}{k-1} \\
&= \frac{1}{k} \cdot \frac{n!}{(k-1)!(n-k+1)!} \binom{n-1}{k-1} \\
&= \frac{1}{k} \binom{n}{k-1} \binom{n-1}{k-1} \\
&= N(n, k).
\end{aligned}$$

□

Proposition 2.4. Consider the generalized Narayana sequence defined by Expression (2.1). Then, given $r > 1$,

$$N_{r-1}(n+1, k+1) = N_r(n, k), \quad (2.4)$$

for $1 \leq k \leq n$.

Proof. We have

$$\begin{aligned}
N_{r-1}(n+1, k+1) &= \frac{1}{n+1-(k+1)+1} \binom{n+1+r-1-1}{k+1+r-1-1} \binom{n+1+r-1-2}{k+1+r-1-2} \\
&= \frac{1}{n-k+1} \binom{n+r-1}{k+r-1} \binom{n+r-2}{k+r-2} \\
&= N_r(n, k).
\end{aligned}$$

□

From the Proposition 2.3 and 2.4, for $1 \leq k \leq n$ the following recurrence follows:

$$\begin{cases} N_1(n, k) = N(n, k); \\ N_r(n, k) = N_{r-1}(n+1, k+1), \quad r > 1. \end{cases} \quad (2.5)$$

In this way, we conclude that

$$\begin{aligned}
N_r(n, k) &= N_{r-1}(n+1, k+1) \\
&= N_{r-2}(n+2, k+2) \\
&\vdots \\
&= N_1(n+r-1, k+r-1) \\
&= N(n+r-1, k+r-1).
\end{aligned}$$

Proposition 2.5. Consider the generalized Narayana sequence defined by Expression (2.1),

$$N_r(n, k) = \binom{n+r-2}{k+r-2} \binom{n+r-1}{k+r-1} - \binom{n+r-2}{k+r-1} \binom{n+r-1}{k+r-2}. \quad (2.6)$$

Proof. It is possible to write Expression (2.1) in the form

$$\begin{aligned}
N_r(n, k) &= \binom{n+r-2}{k+r-2} \binom{n+r-1}{k+r-1} \frac{1}{n-k+1} \\
&= \binom{n+r-2}{k+r-2} \binom{n+r-1}{k+r-1} \left[1 - \frac{(n-k)}{n-k+1} \right] \\
&= \binom{n+r-2}{k+r-2} \binom{n+r-1}{k+r-1} - (n-k) \binom{n+r-2}{k+r-2} \frac{(n+r-1)!}{(k+r-1)!(n-k)!(n-k+1)} \\
&= \binom{n+r-2}{k+r-2} \binom{n+r-1}{k+r-1} - \frac{(n+r-2)!}{(k+r-2)!(n-k-1)!} \frac{(n+r-1)!}{(k+r-1)!(n-k+1)!} \\
&= \binom{n+r-2}{k+r-2} \binom{n+r-1}{k+r-1} - \frac{(n+r-2)!}{(k+r-1)!(n-k-1)!} \frac{(n+r-1)!}{(k+r-2)!(n-k+1)!}
\end{aligned}$$

$$= \binom{n+r-2}{k+r-2} \binom{n+r-1}{k+r-1} - \binom{n+r-2}{k+r-1} \binom{n+r-1}{k+r-2}.$$

□

The case $r = 2$ in the definition given by (2.1) and the Proposition 2.5 are particular case introduced in [5]. Another formula for the generalized Narayana numbers is presented in expression (2.6). Furthermore, it demonstrates that $N_r(n, k)_{1 \leq k \leq n}$ is a sequence of nonnegative integers, given r .

In the tables we describe some examples of generalized Narayana numbers as follows. Below we present the first values for $N_r(n, k)$ with $r = 1, 2, 3, 4$.

Table 2: $N_1(n, k)$

n \ k	1	2	3	4	5	6	7
1	1						
2	1	1					
3	1	3	1				
4	1	6	6	1			
5	1	10	20	10	1		
6	1	15	50	50	15	1	
7	1	21	105	175	105	21	1

Table 3: $N_2(n, k)$

n \ k	1	2	3	4	5	6	7
1	1						
2	3	1					
3	6	6	1				
4	10	20	10	1			
5	15	50	50	15	1		
6	21	105	175	105	21	1	
7	28	196	490	490	196	28	1

Table 4: $N_3(n, k)$

$n \backslash k$	1	2	3	4	5	6	7
1	1						
2	6	1					
3	20	10	1				
4	50	50	15	1			
5	105	175	105	21	1		
6	196	490	490	196	28	1	
7	336	1176	1764	1176	336	36	1

Table 5: $N_4(n, k)$

$n \backslash k$	1	2	3	4	5	6	7
1	1						
2	10	1					
3	50	15	1				
4	175	105	21	1			
5	490	490	196	28	1		
6	1176	1764	1176	336	36	1	
7	2520	5292	5292	2520	540	45	1

Consider the following binomial identity.

Lemma 2.6. For n, k and $r > 1$ non-negative integers, such that $n \neq k - 1$ it is verified

$$\frac{1}{n - k + 1} \binom{n + r - 2}{k + r - 2} = \frac{1}{n + r - 1} \binom{n + r - 1}{n - k + 1}. \quad (2.7)$$

Proof. We have

$$\begin{aligned} \frac{1}{n - k + 1} \binom{n + r - 2}{k + r - 2} &= \frac{1}{(n - k + 1)} \frac{(n + r - 2)!}{(k + r - 2)!(n - k)!} \\ &= \frac{(n + r - 1)(n + r - 2)!}{(n + r - 1)(n - k + 1)(k + r - 2)!(n - k)!} \\ &= \frac{1}{(n + r - 1)} \frac{(n + r - 1)!}{(n - k + 1)!(k + r - 2)!} \end{aligned}$$

$$= \frac{1}{n+r-1} \binom{n+r-1}{n-k+1}.$$

□

Proposition 2.7 establishes another explicit formula for $N_r(n, k)$, and this fact follows immediately from the definition of generalized Narayana sequence and Lemma 2.6.

Proposition 2.7. Given $r \geq 1$ and $1 \leq k \leq n$ it is verified,

$$N_r(n, k) = \frac{1}{n+r-1} \binom{n+r-1}{k+r-1} \binom{n+r-1}{n-k+1}. \quad (2.8)$$

Corollary 2.8. Consider the generalized Narayana sequence defined by Expression (2.1) with $r = 1$ and the relationship between complementary binomial numbers. Then,

$$N_1(n, k) = \frac{1}{n} \binom{n}{k} \binom{n}{k-1},$$

with $1 \leq k \leq n$.

As a particular case, for $r = 2$, we have the corollary below.

Corollary 2.9. Consider the generalized Narayana sequence defined by Expression (2.1) with $r = 2$ and the relationship between complementary binomial numbers. Then,

$$N_2(n, k) = \frac{1}{n+1} \binom{n+1}{k+1} \binom{n+1}{k},$$

with $1 \leq k \leq n$.

The identity of Corollary 2.9 which is nothing more than the closed formula for Narayana numbers (see more details in [5]).

Corollary 2.10. Consider the generalized Narayana sequence defined by expression given by definition (2.1) in case $r = 3$ and the relationship between complementary binomial numbers. Then,

$$N_3(n, k) = \frac{1}{n+2} \binom{n+2}{k+2} \binom{n+2}{k+1},$$

with $1 \leq k \leq n$.

3 The sum of the rows of generalized Narayana sequence

The Catalan numbers form a sequence of natural numbers that occur in various counting problems, often involving recursively defined objects. The name is a reference to the Belgian mathematician Eugéne Charles Catalan (1814-1894), for more details see [3] and [4]. The n -th Catalan number is given in terms of binomial coefficients by $C_n = \frac{1}{n+1} \binom{2n}{n}$, for all $n \geq 0$. In this section, we establish a connection between the Catalan numbers and the sums of the rows of the generalized Narayana numbers. First, consider Equation (2.8). Since it is verified

$$\sum_{k=0}^n N_r(n, k) = \frac{1}{n+r-1} \sum_{k=0}^n \binom{n+r-1}{k+r-1} \binom{n+r-1}{n-k+1}, \quad (3.1)$$

then we are interested in describing a closed formula for the sum

$$\sum_{k=0}^n \binom{n+r-1}{k+r-1} \binom{n+r-1}{n-k+1},$$

We obtain the next formula by replacing $r = 1$ in Equation (3.1) and using the Vandermonde's identity:

$$\begin{aligned} \sum_{k=0}^n N_1(n, k) &= \frac{1}{n} \sum_{k=0}^n \binom{n}{k} \binom{n}{n+1-k} \\ &= \frac{1}{n} \binom{2n}{n+1} \\ &= \frac{(2n)!}{n(n+1)!(n-1)!} \\ &= \frac{1}{n+1} \binom{2n}{n} \\ &= C_n. \end{aligned}$$

In the previous discussion, we established the following proposition.

Proposition 3.1. Consider the generalized Narayana sequence defined by Expression (2.1) with $r = 1$. Then, it is verified

$$\sum_{k=0}^n N_1(n, k) = C_n,$$

where C_n is the n -th Catalan number.

The next lemma will be useful to study the general case.

Lemma 3.2. For n and $r \geq 1$ non-negative integers such as $n + r - 1 \geq 0$, it is verified

$$\sum_{k=0}^n \binom{n+r-1}{r-1+k} \binom{n+r-1}{n+1-k} = \binom{2n+2r-2}{n+r} - \sum_{k=-r+1}^{-1} \binom{n+r-1}{r-1+k} \binom{n+r-1}{n+1-k} \quad (3.2)$$

Proof. From the Binomial Theorem, given n and $r \geq 1$ non-negative integers such as $n + r - 1 \geq 0$, and using equality of polynomials, we have

$$(1+x)^{2n+2r-2} = \sum_{k=0}^{2n+2r-2} \binom{2n+2r-2}{k} x^k. \quad (3.3)$$

On the other hand, we obtain

$$(1+x)^{2n+2r-2} = \sum_{a=0}^{2n+2r-2} \left(\sum_{k=0}^a \binom{n+r-1}{k} \binom{n+r-1}{a-k} \right) x^a. \quad (3.4)$$

Thus, using equality of polynomials, the coefficients of x^{n+r} in (3.3) and (3.4) is equal to

$$\binom{2n+2r-2}{n+r} = \sum_{j=0}^{n+r} \binom{n+r-1}{j} \binom{n+r-1}{n+r-j}, \quad (3.5)$$

where $\binom{a}{b} = 0$, when $a < b$ or a, b negative integers.

Let n and r be integers such as $n + r - 1 \geq 0$, making $j = 0$ then we have $k = -r + 1$ and when $j = n + r$ we obtain $k = n + 1$. Thus, rewriting expression (3.5) we get

$$\begin{aligned} \binom{2n+2r-2}{n+r} &= \sum_{k=-r+1}^{n+1} \binom{n+r-1}{r-1+k} \binom{n+r-1}{n+1-k} \\ &= \sum_{k=-r+1}^{-1} \binom{n+r-1}{r-1+k} \binom{n+r-1}{n+1-k} + \sum_{k=0}^n \binom{n+r-1}{r-1+k} \binom{n+r-1}{n+1-k} \end{aligned}$$

Therefore, we have

$$\sum_{k=0}^n \binom{n+r-1}{r-1+k} \binom{n+r-1}{n+1-k} = \binom{2n+2r-2}{n+r} - \sum_{k=-r+1}^{-1} \binom{n+r-1}{r-1+k} \binom{n+r-1}{n+1-k}.$$

□

Lemma 3.2 will be useful in the proof of the next result which consists of a connection between the Catalan numbers and the generalized Narayana numbers.

Theorem 3.3. *Given $r \geq 2$, consider the generalized Narayana sequence defined by (2.1). Then, it is verified*

$$\sum_{k=0}^n N_r(n, k) = C_{n+r-1} - \frac{1}{n+r-1} \sum_{k=-r+1}^{-1} \binom{n+r-1}{r-1+k} \binom{n+r-1}{n+1-k}, \quad (3.6)$$

where C_n is the n -th Catalan number.

Proof. Since it is verified

$$\begin{aligned} \frac{1}{n+r-1} \binom{2n+2r-2}{n+r} &= \frac{1}{n+r-1} \frac{(2n+2r-2)!}{(n+r)!(n+r-2)!} \\ &= \frac{1}{n+r-1} \frac{(2n+2r-2)!}{(n+r)(n+r-1)!(n+r-2)!} \\ &= \frac{1}{n+r} \frac{(2n+2r-2)!}{[(n+r-1)!]^2} \\ &= C_{n+r-1}, \end{aligned} \quad (3.7)$$

then by replacing Equation (3.7) in Equation (3.2), we get

$$\begin{aligned} \sum_{k=0}^n N_r(n, k) &= \frac{1}{n+r-1} \binom{2n+2r-2}{n+r} - \frac{1}{n+r-1} \sum_{k=-r+1}^{-1} \binom{n+r-1}{r-1+k} \binom{n+r-1}{n+1-k} \\ &= C_{n+r-1} - \frac{1}{n+r-1} \sum_{k=-r+1}^{-1} \binom{n+r-1}{r-1+k} \binom{n+r-1}{n+1-k}. \end{aligned}$$

□

As a consequence of the Theorem 3.3 we have the following Corollaries.

Corollary 3.4. Given $r = 2$, the generalized Narayana sequence defined by (2.1) verify

$$\sum_{k=0}^n N_2(n, k) = C_{n+1},$$

where C_n is the n -th Catalan number.

The Corollary 3.4 is particular result established in [5].

Corollary 3.5. Given $r = 3$, the generalized Narayana sequence defined by (2.1) verify

$$\sum_{k=0}^n N_3(n, k) = C_{n+2} - 1,$$

where C_n is the n -th Catalan number.

4 The central numbers $N_r(2n, n)$

The definition of generalized Narayana numbers given in 2.1, permits us to describe the central generalized Narayana number, given by

$$N_r(2n, n) = \frac{1}{n+1} \binom{2n+r-1}{n+r-1} \binom{2n+r-2}{n+r-2}. \quad (4.1)$$

Now, we will establish some identities between the central generalized Narayana numbers and the Catalan numbers, for cases $r = 1, 2, 3$ and 4.

Proposition 4.1. For $n > 1$ integer, we have

$$N_1(2n, n) = (2n-1)C_n C_{n-1}, \quad (4.2)$$

where C_n is the n -th Catalan number.

Proof. Making $r = 1$ in (4.1) we obtain

$$\begin{aligned} N_1(2n, n) &= \frac{1}{n+1} \binom{2n}{n} \binom{2n-1}{n-1} \\ &= C_n \frac{(2n-1)!}{(n-1)!n!} \\ &= (2n-1)C_n \frac{1}{n} \frac{(2n-2)!}{(n-1)!(n-1)!} \\ &= (2n-1)C_n \frac{1}{n} \binom{2n-2}{n-1} \\ &= (2n-1)C_n C_{n-1}. \end{aligned}$$

□

Proposition 4.2. For n positive integer, we have

$$N_2(2n, n) = (2n + 1)C_n^2, \quad (4.3)$$

where C_n is the n -th Catalan number.

Proof. Making $r = 2$ in (4.1) we obtain

$$\begin{aligned} N_2(2n, n) &= \frac{1}{n+1} \binom{2n+1}{n+1} \binom{2n}{n} \\ &= \frac{1}{n+1} \binom{2n}{n} \binom{2n+1}{n+1} \\ &= C_n \frac{(2n+1)!}{n!(n+1)!} \\ &= C_n \frac{(2n+1)(2n)!}{n!(n+1)n!} \\ &= (2n+1)C_n \frac{1}{(n+1)} \frac{(2n)!}{n!n!} \\ &= (2n+1)C_n C_n \\ &= (2n+1)C_n^2. \end{aligned}$$

□

The property that relates the central number $N_2(2n, n)$ with the Catalan number C_n , obtained in Proposition 4.2, is the same relationship given in [5]. In this case, $N_2(2n, n)$ is odd if and only if n is a Mersene number.

Proposition 4.3. For $n > 1$ integer, it is verified

$$N_3(2n, n) = (2n + 1)C_{n+1}C_n, \quad (4.4)$$

where C_n is the n -th Catalan number.

Proof. Making $r = 3$ in (4.1) we obtain:

$$\begin{aligned}
N_3(2n, n) &= \frac{1}{n+1} \binom{2n+2}{n+2} \binom{2n+1}{n+1} = \frac{1}{n+1} \frac{(2n+1)!}{(n+2)!n!} \frac{(2n+1)!}{(n+1)!n!} \\
&= \frac{1}{n+1} \frac{(2n+2)!}{(n+2)(n+1)!n!} \frac{(2n+1)(2n)!}{(n+1)n!n!} \\
&= \frac{1}{n+2} \frac{(2n+2)!}{(n+1)!(n+1)!} (2n+1) \frac{1}{n+1} \frac{(2n)!}{n!n!} \\
&= C_{n+1}(2n+1)C_n = (2n+1)C_{n+1}C_n.
\end{aligned}$$

□

Proposition 4.4. For n positive integer, it is verified

$$N_4(2n, n) = \frac{1}{2}(n+1)C_{n+2}C_{n+1}, \quad (4.5)$$

where C_n is the n -th Catalan number.

Proof. By replacing $r = 4$ in (4.1) we obtain:

$$\begin{aligned}
N_4(2n, n) &= \frac{1}{n+1} \binom{2n+3}{n+3} \binom{2n+2}{n+2} \\
&= \frac{1}{n+1} \frac{(2n+3)!}{(n+3)!n!} \frac{(2n+2)!}{(n+2)!n!} \\
&= \frac{1}{n+1} \frac{(2n+4)!}{2(n+2)(n+3)(n+2)!n!} \frac{(2n+2)!}{(n+2)(n+1)!n!} \\
&= \frac{(2n+4)!}{2(n+2)(n+3)(n+2)!(n+1)!} \frac{(2n+2)!}{(n+2)(n+1)!n!} \\
&= \frac{1}{2} \frac{1}{n+1} \frac{(2n+4)!}{(n+2)!(n+2)!} \frac{1}{n+2} \frac{(2n+2)!}{(n+1)!(n+1)!} (n+1) \\
&= \frac{1}{2}(n+1)C_{n+1}C_{n+2}
\end{aligned}$$

□

Proposition 4.5. For n, s positive integers, with $s \geq 1$ we have the following identities:

1. $N_{2s+1}(2n, n) = \frac{(2n+2s)(2n+2s-1)}{(n+2s)(n+2s-1)}N_{2s}(2n, n);$
2. $N_{2s}(2n, n) = \frac{(2n+2s-1)(n+s+1)(n+s)!}{(n+1)!n!(n+2s-1)(n+2s-2)^2 \dots (n+s-2)^2(n+s-1)}C_{n+s}C_{n+s-1},$

where C_n is the n -th Catalan number.

Proof. 1. The proof follows directly from the definition given in 2.1 and the following binomial identity $\binom{n}{k} = \frac{n}{k} \binom{n-1}{k-1}$ for $1 \leq k \leq n$.

2. In fact,

$$\begin{aligned}
N_{2s}(2n, n) &= \frac{1}{n+1} \binom{2n+2s-1}{n+2s-1} \binom{2n+2s-2}{n+2s-2} \\
&= \frac{1}{n+1} \frac{(2n+2s-1)!}{(n+2s-1)!n!} \frac{(2n+2s-2)!}{(n+2s-2)!n!} \\
&= \frac{1}{n+1} \frac{(2n+2s)(2n+2s-1)!}{(2n+2s)(n+2s-1)(n+2s-2) \dots (n+s+1)(n+s)!n!} \\
&\quad \frac{(2n+2s-2)!}{(n+2s-2) \dots (n+s-2)(n+s-1)!n!} \\
&= \frac{1}{(2n+2s)(n+1)n!n!} \frac{(2n+2s)!}{(n+2s-1)(n+2s-2) \dots (n+s+1)(n+s)!} \\
&\quad \frac{(2n+2s-2)!}{(n+2s-2)(n+2s-3) \dots (n+s-2)(n+s-1)!} \\
&= \frac{(2n+2s-1)(n+s)!(n+s-1)!(n+s)(n+s+1)}{(2n+2s)(n+1)n![(n+2s-1) \dots (n+s-1)][(n+2s-2) \dots (n+s-2)]} \\
&\quad \frac{1}{(n+s+1)} \binom{2n+2s}{n+s} \frac{1}{n+s} \binom{2n+2s-2}{n+s-1} \\
&= \frac{(2n+2s)(n+s+1)!(n+s)!}{(2n+2s)(n+1)n!(n+2s-1)(n+2s-2)^2 \dots (n+s-2)^2(n+s-1)}C_{n+s}C_{n+s-1}.
\end{aligned}$$

□

5 Conclusions

In this article, we introduce a new sequence using the concept of binomial coefficient in its explicit formula that we call the generalized Narayana numbers, because when we evaluate particular cases we obtain the famous sequence of Narayana Numbers. We have established some properties of the generalized Narayana number.

In addition, some identities are provided. Also, we connect Catalan numbers and generalized Narayana numbers, given some identities involving the elements of these two sequences. Finally, the central generalized Narayana numbers are studied. For some special cases, namely, $r = 1, 2, 3$ and 4 we provided identities involving the central generalized Narayana numbers and Catalan numbers.

It seems that the approaches proposed here are results established and new in the literature. Moreover, this new sequence of numbers can be studied in other aspects, such as their combinatorial interpretation and the connections with the combinatorial interpretation of Catalan numbers.

During the research, it was observed that the generalized Narayana numbers present a great potential for the development of combinatorial theories and interpretations, opening space for the formulation and demonstration of new identities related to the sequence of Catalan numbers. In addition, the relevance of characterizing properties such as divisibility and the identification of prime factors that occur in the sequence of the central terms of these generalized numbers is highlighted.

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