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AN EXPLICIT FORMULA FOR THE STIRLING NUMBERS OF THE FIRST KIND THROUGH INTEGER PARTITIONS

Gabriel de Freitas Pinheiro
Universidade Estadual de Campinas, Brazil.
freitasgabriel688@gmail.com
Irene Magalhães Craveiro
Universidade Federal da Grande Dourados.
irenecraveiro@ufgd.edu.br

Abstract

The aim of this article is to present an explicit formula for the Stirling numbers of the first kind and prove it through counting arguments. This is only possible because these numbers have a strong combinatorial appeal, since we can define them as the number of ways to distribute n people around k identical circular tables, without leaving any empty tables. In order to establish the proof for the main theorem, we will explore some identities involving the Stirling numbers of the first kind with the binomial coefficient, as well as introduce the concept of partitioning positive integers and use it as the main tool for combinatorial arguments in the proof of the main result. Additionally, we prove new identities and others found in the literature through this theorem.

1 Introduction

In the context of generating functions, the Stirling numbers of the first kind are defined by [3] and [7] as the coefficients of x^k in the expansion of the polynomial $x(x+1)\cdots(x+(n-1))$, with $0 < k \leq n$. That is, they represent a numerical sequence generated by this class of polynomials with a fixed degree n, and the coefficients of the powers of x belong to this sequence.

The study of generating functions is primarily attributed to the works of A. De Moivre (1667 - 1754) and was later applied by L. Euler (1707 - 1783) in the field of Additive Number Theory, especially in Partition Theory. This technique allows us to tackle combinatorial problems through algebraic methods, in addition to facilitating the obtainment of solutions for certain classes of recurrences.

In addition to the definition given by [3], we can also read the Stirling numbers of the first kind as the number of permutations of S_n that decompose into exactly k

cycles, as defined by [5] and [6]. However, we will explore the combinatorial concept for these numbers given by [2] when posing the following question: in how many ways can n people sit around k indistinguishable circular tables, without leaving any empty tables?

Definition 1.1. *Let n, k be natural numbers. We call the Stirling numbers of the first kind and denote them by $\left[\begin{matrix} n \\ k \end{matrix}\right]$, the non-negative integers that determine the number of ways to distribute n people among k identical circular tables such that there is at least one person at each table. For theoretical reasons, it is conventionally assumed that $\left[\begin{matrix} 0 \\ 0 \end{matrix}\right] = 1$ and $\left[\begin{matrix} 0 \\ k \end{matrix}\right] = 0$ if $k > 0$.*

2 Binomial coefficients and Stirling numbers of the first kind

In this section, we will establish some identities involving the Stirling numbers of the first kind and the binomial coefficients. We define binomial coefficients combinatorially as the number of ways to form subsets with k elements from a set with n elements. Alternatively, we can define them as the number of ways to choose k objects from a set with n objects.

On the other hand, as previously defined, the Stirling numbers of the first kind represent the number of ways to distribute n people around k identical circular tables such that no table is empty.

That being so, the idea of the results presented in this section is to reinterpret the identities that relate certain classes of the Stirling numbers of the first kind to the binomial coefficient, exploring the combinatorial definition given through the counting of people distributed around identical circular tables. Propositions 2.1, 2.2, 2.3, and 2.4 were demonstrated algebraically by [1], and in the present article, we will validate them using combinatorial arguments.

Proposition 2.1. *For all $n > 1$ we have that $\left[\begin{matrix} n \\ n-1 \end{matrix}\right] = \binom{n}{2}$.*

Proof. If we place 1 person at each table, there will be 1 person left to distribute. As a result, at one of the $n-1$ tables two people will be sitting. We then have $\binom{n}{2}$ ways to make that choice. It follows that:

$$\left[\begin{matrix} n \\ n-1 \end{matrix}\right] = \binom{n}{2}.$$

□

Proposition 2.2. *For all* $n \geq 2$ *we have that* $\left[\begin{array}{c} n \\ n-2 \end{array}\right] = \frac{3n-1}{4}\left(\begin{array}{c} n \\ 3 \end{array}\right)$.

Proof. If we place 1 person at each table, there are 2 people left to distribute. Thus, we have some possibilities for the tables:

1a. one table with 3 people and $n-3$ tables with one person each;

1b. two tables with two people and $n-4$ tables with one person each.

For item 1a, we have $2!\left(\begin{array}{c} n \\ 3 \end{array}\right)$ possibilities. For item 1b, we have

$$\frac{1}{2!}\left(\begin{array}{c} n \\ 2 \end{array}\right)\left(\begin{array}{c} n-2 \\ 2 \end{array}\right).$$

Applying the Additive Principle, we have:

$$\left[\begin{array}{c} n \\ n-2 \end{array}\right] = 2!\left(\begin{array}{c} n \\ 3 \end{array}\right) + \frac{1}{2!}\left(\begin{array}{c} n \\ 2 \end{array}\right)\left(\begin{array}{c} n-2 \\ 2 \end{array}\right).$$

After some manipulations, it follows that:

$$\begin{aligned}
\left[\begin{array}{c} n \\ n-2 \end{array}\right] &= 2!\left(\begin{array}{c} n \\ 3 \end{array}\right) + \frac{1}{2!}\left(\begin{array}{c} n \\ 2 \end{array}\right)\left(\begin{array}{c} n-2 \\ 2 \end{array}\right) = 2!\frac{n!}{3!(n-3)!} + \frac{n!}{(2!)^3(n-4)!} \\
&= \frac{n!(2!)^4}{3!(n-3)!(2!)^3} + \frac{n!3!(n-3)}{(2!)^3 3!(n-3)!} = \frac{(2!)^4}{(2!)^3}\left(\begin{array}{c} n \\ 3 \end{array}\right) + \frac{3!(n-3)}{(2!)^3}\left(\begin{array}{c} n \\ 3 \end{array}\right) \\
&= \frac{6n-18+16}{8}\left(\begin{array}{c} n \\ 3 \end{array}\right) = \frac{6n-2}{8}\left(\begin{array}{c} n \\ 3 \end{array}\right) \\
&= \frac{3n-1}{4}\left(\begin{array}{c} n \\ 3 \end{array}\right).
\end{aligned}$$

□

Proposition 2.3. *For all* $n \geq 3$ *we have* $\left[\begin{array}{c} n \\ n-3 \end{array}\right] = \binom{n}{2}\binom{n}{4}$.

Proof. If we place 1 person at each table, there are 3 people left to distribute. As a consequence, we have some possibilities:

1a. the remaining three people sit together;

1b. two sit together and one sits separately;

1c. all three sit separately.

With these possibilities, we can then have:

2a. one table with 4 people and $n-4$ tables with one person each;

2b. one table with 3 people, another with two people, and $n-5$ tables with one person each;

2c. three tables with two people each and $n-6$ tables with one person each.

For item 2a, we have $\binom{n}{4}$ possibilities for choosing the people, however, since the table is circular, we can permute them in $(4-1)! = 3!$ ways. Hence, we have $\binom{n}{4}3!$ ways to distribute them.

For item 2b, we have $\binom{n}{3}2!$ possibilities for the table with 3 people, for the second table we have $\binom{n-3}{2}$. Hence, by the multiplicative principle, we have

$$\binom{n}{3}\binom{n-3}{2}2!$$

ways to distribute the people according to item 2b.

For item 2c, we have $\frac{1}{3!}\binom{n}{2}\binom{n-2}{2}\binom{n-4}{2}$ ways to distribute the people around the $n-3$ circular tables.

Thus, by the Additive Principle, the number of possibilities to distribute n people around $n-3$ identical circular tables without leaving any empty is:

$$\left[\begin{matrix} n \\ n-3 \end{matrix}\right] = \binom{n}{4}3! + \binom{n}{3}\binom{n-3}{2}2! + \frac{1}{3!}\binom{n}{2}\binom{n-2}{2}\binom{n-4}{2}.$$

Let's manipulate the previous equation:

$$\begin{aligned}
\left[\begin{array}{c} n \\ n-3 \end{array}\right] &= \frac{n!}{4!(n-4)!}3! + \frac{n!}{2!3!(n-5)!}2! + \frac{n!}{(2!)^3 3!(n-6)!} \\
&= \frac{n!}{4!(n-4)!}3! + \frac{n!(2!)^2(n-4)}{4!(n-4)!} + \frac{n!(n-5)(n-4)}{2.4!(n-4)!} \\
&= 3!\binom{n}{4} + (2!)^2(n-4)\binom{n}{4} + \frac{(n-5)(n-4)}{2}\binom{n}{4} \\
&= \frac{n^2-9n+20+2^3(n-4)+12}{2}\binom{n}{4} \\
&= \frac{n^2-9n+20+8n-32+12}{2}\binom{n}{4} \\
&= \frac{n^2-n}{2}\binom{n}{4} = \frac{n(n-1)}{2}\binom{n}{4} \\
&= \binom{n}{2}\binom{n}{4}.
\end{aligned}$$

□

In [1], we proved Propositions 2.1, 2.2, and 2.3 using algebraic arguments, using both the Principle of Finite Induction and Stifel's Relation, as well as validating the results by applying Girard's relations to the generating function for the Stirling numbers of the first kind. In this section, we chose to prove them using counting tools, as they provide a combinatorial argument that will be necessary for the next section.

Proposition 2.4 was not presented in [1], but we decided to address it in this work to thoroughly explore the combinatorial reasoning involving the Stirling numbers of the first kind.

Proposition 2.4. *For all* $n \geq 4$*, we have that* $\left[\begin{array}{c} n \\ n-4 \end{array}\right] = \frac{15n^3-30n^2+5n+2}{48}\binom{n}{5}$.

Proof. If we place 1 person at each table, there are 4 people left to be allocated. Below are the possibilities for the number of people at each table:

1a. one table with 5 people and $n-5$ tables with one person each;

1b. one table with 4 people, another with 2 people, and $n-6$ tables with 1 person each;

1c. two tables with 3 people each and $n-6$ tables with 1 person each;

1d. one table with 3 people and two tables with 2 people each, and $n-7$ tables with 1 person each;

1e. four tables with 2 people each and $n-8$ tables with 1 person each.

It follows that:

$$\begin{bmatrix} n \\ n-4 \end{bmatrix} = 4!\binom{n}{5} + 3!\binom{n}{4}\binom{n-4}{2} + \frac{1}{2!}(2!)^2\binom{n}{3}\binom{n-3}{3} + \frac{1}{2!}2!\binom{n}{3}\binom{n-3}{2}\binom{n-5}{2} + \frac{1}{4!}\binom{n}{2}\binom{n-2}{2}\binom{n-4}{2}\binom{n-6}{2}.$$

After some manipulations, we have:

$$\begin{aligned}
\begin{bmatrix} n \\ n-4 \end{bmatrix} &= 4!\binom{n}{5} + \frac{1}{8}\frac{n!}{(n-6)!} + \frac{1}{18}\frac{n!}{(n-6)!} + \frac{1}{4!}\frac{n!}{(n-7)!} + \frac{1}{4!(2!)^4}\frac{n!}{(n-8)!} \\
&= 4!\binom{n}{5} + \frac{13}{72}\frac{n!}{(n-6)!} + \frac{1}{4!}\frac{n!}{(n-7)!} + \frac{1}{4!(2!)^4}\frac{n!}{(n-8)!} \\
&= 4!\binom{n}{5} + 5!\frac{13}{72}(n-5)\binom{n}{5} + 5!\frac{1}{4!}(n-6)(n-5)\binom{n}{5} + \\
&\quad \frac{5!}{4!(2!)^4}(n-7)(n-6)(n-5)\binom{n}{5} \\
&= \frac{1152 + 1040(n-5) + 240(n-6)(n-5) + 15(n-7)(n-6)(n-5)}{48}\binom{n}{5} \\
&= \frac{15n^3 - 30n^2 + 5n + 2}{48}\binom{n}{5}.
\end{aligned}$$

□

Note that in each of the propositions in this section, we followed this reasoning: first, we placed 1 person at each of the $n-k$ tables, with $k = 1, ..., 4$. As a result, out of n people, k people remain to be distributed. In other words, we need to partition the positive integer k to determine the possible distributions.

With this idea in mind, we will present an explicit formula for the Stirling numbers of the first kind in the next section, where, through the concept of integer partitions, we will provide a proof using combinatorial arguments for this formula.

3 Integer partitions and Stirling numbers of the first kind

According to [4], a *partition of a positive integer* n is a collection of positive integers whose sum is n. The formalization of this concept is given in Definition 3.1.

Definition 3.1. *Let $n > 0$. An integer partition of n is a sequence of positive integers $(\lambda_1, \lambda_2, \cdots, \lambda_t)$ such that*

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_t \textbf{ and } \lambda_1 + \lambda_2 \cdots + \lambda_t = n.$$

The λ_i's are called the parts of the partition, with $i = 1, \cdots, t$ and t represents the number of parts in the partition. Clearly, $t \leq n$.

n	3	4	5	6	7
Partitions of n	3 2+1 1+1+1	4 3+1 2+2 2+1+1 1+1+1+1	5 4+1 3+2 3+1+1 2+2+1 2+1+1+1 1+1+1+1+1	6 5+1 4+2 4+1+1, 3+3 3+2+1 3+1+1+1 2+2+2 2+2+ 1+1 2+ 1+1 +1+1 1+1+1+1+1+1	7 6+1 5+2 5+1+1 4+3 4+2+1 4+1+1+1 3+3+1 3+2+2 3+2+1+1 3+1+1+1+1 2+2+2+1 2+2+1+1+1 2+ 1+ 1+1+1+1 1+1+1+1+1+1+1
$p(n)$	3	5	7	11	15

Table 1: Partitions for $n = 3, 4, 5, 6$ e 7

In Table 1, we present all the partitions of the numbers 3, 4, 5, 6, and 7. Furthermore, since $p(n)$ represents the number of partitions of the positive integer n, we have $p(3) = 3$, $p(4) = 5$, $p(5) = 7$, $p(6) = 11$, and $p(7) = 15$. One thing to note is that $p(n)$ grows rapidly, exhibiting exponential behavior.

Because of this, many mathematicians have dedicated themselves to the search for an explicit formula for the function $p(n)$. It was thanks to the influential contributions of S. Ramanujan, G.H. Hardy, and H. Radamacher that an asymptotic expression for $p(n)$ was developed. This expression is represented by $\frac{1}{4n\sqrt{3}}e^{\pi\sqrt{\frac{2n}{3}}}$ as n tends to infinity, revealing the asymptotic behavior of the function that describes the number of partitions of a positive integer n.

Consider $\pi = \alpha_1\lambda_1 + \alpha_2\lambda_2 + \cdots + \alpha_t\lambda_t$, where the α_i and λ_i are positive integers for all $i = 1, \cdots, t$. From definition 3.1, we have that π represents a partition of a positive integer, but with a different configuration. The α_i's tell us how many times the part λ_i appears in the partition.

For example, consider $\pi = 3 + 2 + 2 + 1 + 1 + 1$, a partition of the number 10. Note that the part 3 appears only once, the part 2 appears twice, and the part 1 appears three times.

With the concepts defined so far, we state in Theorem 3.2 the main result of this article. From the partitions of the positive integer k, we establish how to distribute people around the circular tables. The proof of Theorem 3.2 is done combinatorially, so the use of binomial numbers and the Additive and Multiplicative Principles will be recurrent. Consider $p(k, n-k)$ as the partitions of k where the number of parts does not exceed $n-k$.

Theorem 3.2. *Let* $\pi_i = \sum_{j=1}^{t} \alpha_i^{(j)}\lambda_i^{(j)}$ *be a partition of* k *with* $t \leq k$, $\lambda_i^{(1)} \leq \lambda_i^{(2)} \leq \cdots \leq \lambda_i^{(t)}$ *and* $i = 1, \cdots, p(k, n-k)$. *Then, for a fixed* k *we have:*

$$\left[\begin{array}{c} n \\ n-k \end{array}\right] = \sum_{i=1}^{p(k,n-k)} \prod_{j=1}^{t} \frac{1}{\alpha_i^{(j)}!} \frac{\left(n - \sum_{r=1}^{j-1} \alpha_i^{(r)}(1+\lambda_i^{(r)})\right)!}{(1+\lambda_i^{(j)})^{\alpha_i^{(j)}} \left(n - \sum_{r=1}^{j} \alpha_i^{(r)}(1+\lambda_i^{(r)})\right)!}$$

Proof. We have $n-k$ identical circular tables. If we place 1 person at each table, there are k people left to be distributed. There are $p(k, n-k)$ ways to allocate these remaining k people.

Consider $\pi_1, \pi_2, \cdots, \pi_{p(k,n-k)}$ as the partitions of k where the number of parts does not exceed $n-k$, where $\pi_i = \sum_{j=1}^{t} \alpha_i^{(j)}\lambda_i^{(j)}$, with $i = 1, \cdots, p(k, n-k)$ and $\alpha_i^{(j)}$ representing how many times the part $\lambda_i^{(j)}$ appears in the partition.

For a given i, the number of people at each table is: $\alpha_i^{(1)}$ tables with $1+\lambda_i^1$ people, $\alpha_i^{(2)}$ tables with $1+\lambda_i^{(2)}$ people, and so on, up to $\alpha_i^{(t)}$ tables with $1+\lambda_i^t$ people. The remaining $n-k-(\alpha_i^{(1)}+\alpha_i^{(2)}+\cdots+\alpha_i^{(t)})$ tables have 1 person each.

For any i, if $j=1$, then there should be $1+\lambda_i^{(1)}$ people at $\alpha_i^{(1)}$ tables, then the number of choices is:

$$\begin{aligned}
\left[\begin{array}{c} n \\ n-k \end{array}\right]_i^{(1)} &= \frac{(\lambda_i^{(1)}!)^{\alpha_i^{(1)}}}{\alpha_i^{(1)}!}\binom{n}{1+\lambda_i^{(1)}}\binom{n-(1+\lambda_i^{(1)})}{1+\lambda_i^{(1)}}\cdots\binom{n-(\alpha_i^{(1)}-1)(1+\lambda_i^{(1)})}{1+\lambda_i^{(1)}} \\
&= \frac{(\lambda_i^{(1)}!)^{\alpha_i^{(1)}}}{\alpha_i^{(1)}!}\frac{n!}{[(1+\lambda_i^{(1)})!]^{\alpha_i^{(1)}}(n-\alpha_i^{(1)}(1+\lambda_i^{(1)}))!} \\
&= \frac{1}{\alpha_i^{(1)}!}\frac{n!}{(1+\lambda_i^{(1)})^{\alpha_i^{(1)}}(n-\alpha_i^{(1)}(1+\lambda_i^{(1)}))!}.
\end{aligned}$$

For $j=2$, we must remember that we no longer have n left, but $n-\alpha_i^{(1)}(1+\lambda_i^{(1)})$. Consequently, for any j, $j=1,\cdots,t$, we have:

$$\begin{aligned}
\left[\begin{array}{c} n \\ n-k \end{array}\right]_i^{(j)} &= \frac{(\lambda_i^{(j)}!)^{\alpha_i^{(j)}}}{\alpha_i^{(j)}!}\left(\begin{array}{c} n-\displaystyle\sum_{r=1}^{j-1}\alpha_i^{(r)}(1+\lambda_i^{(r)}) \\ 1+\lambda_i^{(j)} \end{array}\right)\left(\begin{array}{c} n-\displaystyle\sum_{r=1}^{j-1}\alpha_i^{(r)}(1+\lambda_i^{(r)})-(1+\lambda_i^{(j)}) \\ 1+\lambda_i^{(j)} \end{array}\right)\cdots \\
&\quad \left(\begin{array}{c} n-\displaystyle\sum_{r=1}^{j-1}\alpha_i^{(r)}(1+\lambda_i^{(r)})-(\alpha_i^{(j)}-1)(1+\lambda_i^{(j)}) \\ 1+\lambda_i^{(j)} \end{array}\right) \\
&= \frac{(\lambda_i^{(j)}!)^{\alpha_i^{(j)}}}{\alpha_i^{(j)}!}\frac{\left(n-\displaystyle\sum_{r=1}^{j-1}\alpha_i^{(r)}(1+\lambda_i^{(r)})\right)!}{[(1+\lambda_i^{(j)})!]^{\alpha_i^{(j)}}\left(n-\displaystyle\sum_{r=1}^{j-1}\alpha_i^{(r)}(1+\lambda_i^{(r)})-\alpha_i^{(j)}(1+\lambda_i^{(j)})\right)!} \\
&= \frac{1}{\alpha_i^{(j)}!}\frac{\left(n-\displaystyle\sum_{r=1}^{j-1}\alpha_i^{(r)}(1+\lambda_i^{(r)})\right)!}{(1+\lambda_i^{(j)})^{\alpha_i^{(j)}}\left(n-\displaystyle\sum_{r=1}^{j}\alpha_i^{(r)}(1+\lambda_i^{(r)})\right)!}.
\end{aligned}$$

Thus, for each $i\in[p(k,n-k)]$, we have:

$$\left[\begin{array}{c} n \\ n-k \end{array}\right]_i = \prod_{j=1}^{t} \frac{1}{\alpha_i^{(j)}!} \frac{\left(n - \sum_{r=1}^{j-1} \alpha_i^{(r)}(1+\lambda_i^{(r)})\right)!}{(1+\lambda_i^{(j)})^{\alpha_i^{(j)}} \left(n - \sum_{r=1}^{j} \alpha_i^{(r)}(1+\lambda_i^{(r)})\right)!}.$$

It follows, by the additive principle, that:

$$\left[\begin{array}{c} n \\ n-k \end{array}\right] = \sum_{i=1}^{p(k,n-k)} \prod_{j=1}^{t} \frac{1}{\alpha_i^{(j)}!} \frac{\left(n - \sum_{r=1}^{j-1} \alpha_i^{(r)}(1+\lambda_i^{(r)})\right)!}{(1+\lambda_i^{(j)})^{\alpha_i^{(j)}} \left(n - \sum_{r=1}^{j} \alpha_i^{(r)}(1+\lambda_i^{(r)})\right)!}.$$

□

Note that if $s = n - k$, then $\left[\begin{array}{c} n \\ s \end{array}\right]$ is also determined by Theorem 3.2. Below we present some examples of classes of the Stirling numbers of the first kind in order to apply Theorem 3.2.

Example 3.3. Consider $n = 7$ and $k = 5$ and let's use Theorem 3.2 to deduce a formula for $\left[\begin{array}{c} 7 \\ 7-5 \end{array}\right] = \left[\begin{array}{c} 7 \\ 2 \end{array}\right]$. From Table 1 we have that the partitions of 5 into a maximum of 2 parts are: $5, 4+1, 3+2$. Thus,

- $\pi_1 = \alpha_1^1\lambda_1^1 + \alpha_1^2\lambda_1^2 = 1 \times 1 + 1 \times 4;$
- $\pi_2 = \alpha_2^1\lambda_2^1 + \alpha_2^2\lambda_2^2 = 1 \times 2 + 1 \times 3;$
- $\pi_3 = \alpha_3^1\lambda_3^1 = 1 \times 5;$

Then,

$$i = 1 \quad \rightarrow \quad \frac{1}{1!}\frac{7!}{(1+1)^1(7-1(1+1))!}\frac{1}{1!}\frac{(7-1(1+1))!}{(1+4)^1(7-1(1+1)-1(1+4))!} = \frac{7!}{2(7-2)!}\frac{(7-2)!}{5(7-2-5)!} = \frac{7!}{10(7-7)!} = \frac{7!}{10} = 504;$$

$$
\begin{aligned}
i = 2 \quad \rightarrow \quad & \frac{1}{1!}\frac{7!}{(1+2)^1(7-1(1+2))!}\frac{1}{1!}\frac{(7-1(1+2))!}{(1+3)^1(7-1(1+2)-1(1+3))!} = \\
& \frac{7!}{3(7-3)!}\frac{(7-3)!}{4(7-3-4)!} = \frac{7!}{12(7-7)!} = \frac{7!}{12} = 420; \\
i = 5 \quad \rightarrow \quad & \frac{1}{1!}\frac{7!}{(1+5)^1(7-1(1+5))!} = \frac{7!}{6(7-6)!} = \frac{7!}{6} = 840.
\end{aligned}
$$

Thus, we have:

$$
\begin{bmatrix} 7 \\ 2 \end{bmatrix} = 504 + 420 + 840 = 1764.
$$

Example 3.4. Let's find $\begin{bmatrix} n \\ n-3 \end{bmatrix}$ using the formula given by Theorem 3.2. If we place 1 person at each table, we have $n-(n-3) = 3$ people left. We can partition the number 3 in three ways: $3, 2+1, 1+1+1$. Note that: $3 = 1\times 3, 2+1 = 1\times 2+1\times 1, 1+1+1 = 3\times 1$. Then, we have the following possibilities:

- 1 table with $3+1$ people and $n-4$ tables with 1 person;
- 1 table with $2+1$ people, 1 table with $1+1$ people and $n-5$ tables with 1 person;
- 3 tables with $1+1$ people and $n-6$ tables with 1 person.

Then, putting it into the formula structure, we have the following data:

- $\pi_1 = \alpha_1^{(1)}\lambda_1^{(1)} = 1 \times 3$;
- $\pi_2 = \alpha_2^{(1)}\lambda_2^{(1)} + \alpha_2^{(2)}\lambda_2^{(2)} = 1 \times 2 + 1 \times 1$;
- $\pi_3 = \alpha_3^{(1)}\lambda_3^{(1)} = 3 \times 1$;

So, it follows that:

$$
\begin{aligned}
i = 1 \quad \rightarrow \quad & \frac{1}{1!}\frac{n!}{(1+3)^1(n-1(1+3))!} = \frac{n!}{4(n-4)!}; \\
i = 2 \quad \rightarrow \quad & \frac{1}{1!}\frac{n!}{3(n-1(1+2))!}\frac{(n-1(1+2))!}{(1+1)^1(n-1(1+2)-1(1+1))!} = \frac{n!}{3(n-3)!}\frac{(n-3)!}{2(n-5)!} \\
& = \frac{n!}{6(n-5)!}; \\
i = 3 \quad \rightarrow \quad & \frac{1}{3!}\frac{n!}{(1+1)^3(n-3(1+1))!} = \frac{n!}{3!2^3(n-6)!};
\end{aligned}
$$

Hence, we have:

$$\begin{aligned}\left[\begin{array}{c} n \\ n-3 \end{array}\right] &= \frac{n!}{4(n-4)!}+\frac{n!}{6(n-5)!}+\frac{n!}{48(n-6)!}\\ &= 3!\binom{n}{4}+4(n-4)\binom{n}{4}+\frac{1}{2}(n-5)(n-4)\binom{n}{4}\\ &= \frac{12+8n-32+n^2-9n+20}{2}\binom{n}{4}\\ &= \frac{n^2-n}{2}\binom{n}{4}=\binom{n}{2}\binom{n}{4}.\end{aligned}$$

4 Applications of the Theorem 3.2

The corollary 4.1 is a result that appear in the literature, however the proofs is different from what we will do in this article, as they follow directly from the main theorem that uses the concept of partition.

Corollary 4.1. *Let n be an integer such that $n > 4$. Then*

$$\left[\begin{array}{c} n \\ n-4 \end{array}\right] = \frac{15n^3-30n^2+5n+2}{48}\binom{n}{5}.$$

Proof. Consider $k = 4$ and let's use theorem 3.2 to deduce a formula for $\left[\begin{array}{c} n \\ n-4 \end{array}\right]$. From Table 1 we have that the partitions of the number 4 are: $4, 3+1, 2+2, 2+1+1, 1+1+1+1$. Thus,

- $\pi_1 = \alpha_1^1\lambda_1^1 = 1\times 4$;
- $\pi_2 = \alpha_2^1\lambda_2^1 + \alpha_2^2\lambda_2^2 = 1\times 1 + 1\times 3$;
- $\pi_3 = \alpha_3^1\lambda_3^1 = 2\times 2$;
- $\pi_4 = \alpha_4^1\lambda_4^1 + \alpha_4^2\lambda_4^2 = 2\times 1 + 1\times 2$;
- $\pi_5 = \alpha_5^1\lambda_5^1 = 4\times 1$.

Then,

$$\begin{aligned}
i=1 &\;\to\; \frac{1}{1!}\frac{n!}{(1+4)^1(n-1(1+4))!}=\frac{n!}{5(n-5)!};\\
i=2 &\;\to\; \frac{1}{1!}\frac{n!}{(1+1)^1(n-1(1+1))!}\frac{1}{1!}\frac{(n-1(1+1))!}{(1+3)^1(n-1(1+1)-1(1+3))!}=\\
&\quad\;\frac{n!}{2(n-2)!}\frac{(n-2)!}{4(n-6)!}=\frac{n!}{8(n-6)!};
\end{aligned}$$

$$\begin{aligned}
i=3 &\;\to\; \frac{1}{2!}\frac{n!}{(1+2)^2(n-2(1+2))!}=\frac{n!}{6\cdot 3(n-6)!};\\
i=4 &\;\to\; \frac{1}{2!}\frac{n!}{(1+1)^2(n-2(1+1))!}\frac{1}{1!}\frac{(n-2(1+1))!}{(1+2)^1(n-2(1+1)-1(1+2))!}=\\
&\quad\;\frac{n!}{8(n-4)!}\frac{(n-4)!}{3(n-7)!}=\frac{n!}{24(n-7)!};\\
i=5 &\;\to\; \frac{1}{4!}\frac{n!}{(1+1)^4(n-4(1+1))!}=\frac{n!}{24\cdot 16(n-8)!};
\end{aligned}$$

Thus, we have:

$$\begin{aligned}
\left[\begin{array}{c} n \\ n-4 \end{array}\right] &= \frac{n!}{5(n-5)!}+\frac{n!}{8(n-6)!}+\frac{n!}{6\cdot 3(n-6)!}+\frac{n!}{24(n-7)!}+\frac{n!}{24\cdot 16(n-8)!}\\
&= \binom{n}{5}\Big[4!+15(n-5)+\frac{20}{3}(n-5)+5(n-6)(n-5)+\\
&\qquad \frac{5}{16}(n-7)(n-6)(n-5)\Big]\\
&= \frac{15n^3-30n^2+5n+2}{48}\binom{n}{5}
\end{aligned}$$

□

We did not find the next result in the literature, so we decided to address it in this section.

Corollary 4.2. *Let n be an integer such that $n>5$. Then*

$$\left[\begin{array}{c} n \\ n-5 \end{array}\right]=\frac{3n^4-10n^3+5n^2+2n}{16}\binom{n}{6}.$$

Proof. Let's determine an expression for $\begin{bmatrix} n \\ n-5 \end{bmatrix}$ using Theorem 3.2. From Table 1, we have that the partitions of the number 5 are $\pi_1 = 5; \pi_2 = 4+1$, $\pi_3 = 3+2$, $\pi_4 = 3+1+1$, $\pi_5 = 2+2+1$, $\pi_6 = 2+1+1+1$, $\pi_7 = 1+1+1+1+1$. So,

- $\pi_1 = \alpha_1^{(1)}\lambda_1^{(1)} = 1 \times 5;$
- $\pi_2 = \alpha_2^{(1)}\lambda_2^{(1)} + \alpha_2^{(2)}\lambda_2^{(2)} = 1 \times 1 + 1 \times 4;$
- $\pi_3 = \alpha_3^{(1)}\lambda_3^{(1)} + \alpha_3^{(2)}\lambda_3^{(2)} = 1 \times 2 + 1 \times 3;$
- $\pi_4 = \alpha_4^{(1)}\lambda_4^{(1)} + \alpha_4^{(2)}\lambda_4^{(2)} = 2 \times 1 + 1 \times 3;$
- $\pi_5 = \alpha_5^{(1)}\lambda_5^{(1)} + \alpha_5^{(2)}\lambda_5^{(2)} = 1 \times 1 + 2 \times 2.$
- $\pi_6 = \alpha_6^{(1)}\lambda_6^{(1)} + \alpha_6^{(2)}\lambda_6^{(2)} = 3 \times 1 + 1 \times 2;$
- $\pi_7 = \alpha_7^{(1)}\lambda_7^{(1)} = 5 \times 1.$

As a result,

$$
\begin{aligned}
i = 1 \quad &\rightarrow \quad \frac{1}{1!}\frac{n!}{(1+5)^1(n-1(1+5))!} = \frac{n!}{6(n-6)!};\\
i = 2 \quad &\rightarrow \quad \frac{1}{1!}\frac{n!}{(1+1)^1(n-1(1+1))!}\frac{1}{1!}\frac{(n-1(1+1))!}{(1+4)^1(n-1(1+1)-1(1+4))!} =\\
&\frac{n!}{2(n-2)!}\frac{(n-2)!}{5(n-7)!} = \frac{n!}{10(n-7)!};\\
i = 3 \quad &\rightarrow \quad \frac{1}{1!}\frac{n!}{(1+2)^1(n-1(1+2))!}\frac{1}{1!}\frac{(n-1(1+2))!}{(1+3)^1(n-1(1+2)-1(1+3))!} =\\
&\frac{n!}{3(n-3)!}\frac{(n-3)!}{4(n-7)!} = \frac{n!}{12(n-7)!};\\
i = 4 \quad &\rightarrow \quad \frac{1}{2!}\frac{n!}{(1+1)^2(n-2(1+1))!}\frac{1}{1!}\frac{(n-2(1+1))!}{(1+3)^1(n-2(1+1)-1(1+3))!} = \frac{n!}{32(n-8)!};\\
i = 5 \quad &\rightarrow \quad \frac{1}{1!}\frac{n!}{(1+1)^1(n-1(1+1))!}\frac{1}{2!}\frac{(n-1(1+1))!}{(1+2)^2(n-1(1+1)-2(1+2))!} =\\
&\frac{n!}{2(n-2)!}\frac{(n-2)!}{18(n-8)!} = \frac{n!}{36(n-8)!};
\end{aligned}
$$

$$i = 6 \quad \rightarrow \quad \frac{1}{3!}\frac{n!}{(1+1)^3(n-3(1+1))!}\frac{(n-3(1+1))!}{(1+2)^1(n-3(1+1)-1(1+2))!} = \frac{n!}{144(n-9)!};$$
$$i = 7 \quad \rightarrow \quad \frac{1}{5!}\frac{n!}{(1+1)^5(n-5(1+1))!} = \frac{n!}{5!\cdot 2^5(n-10)!};$$

After some operations, we find that:

$$\begin{aligned}\left[\begin{matrix} n \\ n-5 \end{matrix}\right] &= 5!\binom{n}{6} + 132(n-6)\binom{n}{6} + \frac{85}{2}(n-6)(n-7)\binom{n}{6} + \\ &\quad 5(n-6)(n-7)(n-8)\binom{n}{6} + \frac{3}{16}(n-6)(n-7)(n-8)(n-9)\binom{n}{6}.\end{aligned}$$

Then, by expanding the products and finding the lowest common multiple, we have:

$$\left[\begin{matrix} n \\ n-5 \end{matrix}\right] = \frac{3n^4 - 10n^3 + 5n^2 + 2n}{16}\binom{n}{6}.$$

□

In [1], the following identity for the Stirling numbers of the first kind is given: Let n and k be positive integers with $n > k$, then

$$\left[\begin{matrix} n \\ n-k \end{matrix}\right] = \sum_{0 \le i_1 < i_2 < \cdots < i_k} i_1 i_2 \cdots i_k \tag{4.1}$$

with $i_1, i_2, \cdots, i_k \in \{1, 2, \cdots, n-1\}$.

That is, the authors relate the Stirling numbers of the first kind with the sum of all products of k distinct factors, where these factors belong to the set $\{1, 2, \cdots, n-1\}$. Thus, from Equation (4.1) and Corollaries 4.1 and 4.2, it follows that:

1)
$$\frac{15n^3 - 30n^2 + 5n + 2}{48}\binom{n}{5} = \left[\begin{matrix} n \\ n-4 \end{matrix}\right] = \sum_{0 \le i_1 < i_2 i_3 < i_4} i_1 i_2 i_3 i_4,$$

with $i_1, i_2, i_3, i_4 \in \{1, 2, \cdots, n-1\}$;

2)
$$\frac{3n^4 - 10n^3 + 5n^2 + 2n}{16}\binom{n}{6} = \left[\begin{matrix} n \\ n-5 \end{matrix}\right] = \sum_{0 \le i_1 < i_2 i_3 < i_4 < i_5} i_1 i_2 i_3 i_4 i_5,$$

with $i_1, i_2, i_3, i_4, i_5 \in \{1, 2, \cdots, n-1\}$.

As a consequence of Theorem 3.2 and Theorem 4.2 [1], the next result follows.

Corollary 4.3. *Let* $\pi_i = \sum_{j=1}^{t} \alpha_i^{(j)} \lambda_i^{(j)}$ *be a partition of* k *with* $t \leq k$, $\lambda_i^{(1)} \leq \lambda_i^{(2)} \leq \cdots \leq \lambda_i^{(t)}$ *and* $i = 1, \cdots, p(k, n-k)$. *Then, for a fixed* k, *we have:*

$$\sum_{0 \leq i_1 < i_2 < \cdots < i_k} i_1 i_2 \cdots i_k = \sum_{i=1}^{p(k,n-k)} \prod_{j=1}^{t} \frac{1}{\alpha_i^{(j)}!} \frac{\left(n - \sum_{r=1}^{j-1} \alpha_i^{(r)}(1 + \lambda_i^{(r)})\right)!}{(1 + \lambda_i^{(j)})^{\alpha_i^{(j)}} \left(n - \sum_{r=1}^{j} \alpha_i^{(r)}(1 + \lambda_i^{(r)})\right)!}$$

with $i_1, i_2, \cdots, i_k \in \{1, 2, \cdots, n-1\}$.

Thus, this result shows us that there is a direct relationship between the sum of all products of k different factors, where these factors belong to the set $\{1, 2, \cdots, n-1\}$ and the partitions of k into at most $n-k$ parts.

5 Final Considerations

In this article, we explored some identities involving the Stirling numbers of the first kind with the binomial coefficient. However, unlike previous works, we opted for demonstrations using combinatorial arguments.

Furthermore, we explored some concepts related to Partition Theory, which is a topic of great importance for enunciating the main result of this article. Some modifications in the definition of partition allowed us to visualize a better way to distribute people around identical circular tables, enabling the development of a purely combinatorial proof of the theorem.

Some articles in the references provide proofs for identities involving the Stirling numbers of the first kind analytically or using other means, such as Girard's relations. Others use the recurrence relation that these numbers possess and, through the Principle of finite induction, carry out some demonstrations.

The idea of this article was to explore the concept of integer partition to present an explicit formula for the Stirling numbers of the first kind and, based on the concepts defined throughout the text, present a combinatorial proof for the main theorem.

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