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ON COMBINATORIAL IDENTITIES FOR R-GENERALIZED FIBONACCI SEQUENCES

Reginaldo Leoncio Silva
Universidade Estadual do Sudoeste da Bahia, Brazil.
reggekant@yahoo.com.br
Elen Viviani Pereira Spreafico
Universidade Federal do Mato Grosso do Sul, Brazil.
elen.spreafico@ufms.br

Abstract

In this paper, we investigate combinatorial identities for $r-$generalized Fibonacci sequences. For this purpose, we established a combinatorial fundamental system related to the sequences of $r-$generalized Fibonacci type, and using the properties of the Casoratian matrix associated we obtain new combinatorial identities. Moreover, some special cases are studied and new general combinatorial identities are provided for these special sequences of numbers.

Keywords: Fundamental system, properties, combinatorial identities.

1 Introduction

The classical Fibonacci $\{F_n\}_{n\geq 0}$ and Pell $\{P_n\}_{n\geq 0}$ sequence are the recursive relations given by $F_{n+1} = F_n + F_{n-1}$, $\forall n \geq 1$, where the initial conditions are $F_0 = 0$ and $F_1 = 1$ and $P_{n+1} = 2P_n + P_{n-1}$, for $n \geq 1$, with the initial conditions $P_0 = 0$, $P_1 = 1$, respectively. These sequences and their generalizations are widely studied from both algebraic, analytic, combinatorial and matrix perspective, and it is an interesting subject of several important properties and identities (see, for example, [1], [4], [10], [12], [15], [16]). One of these generalizations concerns the so-called weighted $r-$generalized Fibonacci sequences, defined as follows. The weighted $r-$generalized Fibonacci sequences $\{v_n\}_{n\geq 0}$ of higher order $r \geq 2$ is defined by giving two sequences $\{a_j\}_{0\leq j\leq r-1}$ and $\{\alpha_j\}_{0\leq j\leq r-1}$ $(r \geq 2)$ of $\mathbb{K}$ (field of real or complex numbers), in the form,

$$v_n = \sum_{i=0}^{r-1} a_i v_{n-i-1} \quad \text{for} \quad n \geq r, \tag{1.1}$$

$$v_n = \alpha_n \quad \text{for} \quad n = 0,\ 1,\ ...,\ r-1. \tag{1.2}$$

The sequences $\{a_j\}_{0\leq j\leq r-1}$ and $\{\alpha_j\}_{0\leq j\leq r-1}$ are known in the literature as the *coefficients* and the *initial conditions* (or *initial data*) of the sequence (1.1). For reasons of simplicity, in the sequel we will refer to these sequences by sequences (1.1). When considering only Expression (1.1), without taking into account the initial conditions (1.2), we are then dealing with a linear difference equation of constant coefficients.

Let $a_0, a_1, \cdots, a_{r-1}$ be real or complex numbers and consider the combinatorial expression defined in [13],

$$\rho(n,r) = \sum_{k_0+k_1+\cdots+rk_{r-1}=n-r} \frac{(k_0+\cdots+k_{r-1})!}{k_0!k_1!\cdots k_{r-1}!} a_0^{k_0}.a_1^{k_1}\cdots a_{r-1}^{k_{r-1}},$$

with $n \geq r$, where $\rho(r,r) = 1$ and $\rho(n,r) = 0$ for $0 \leq n \leq r-1$. It was showed that $\rho(n,r)$ satisfies the following linear difference equation

$$\rho(n+1,r) = a_0\rho(n,r) + a_1\rho(n-1,r) + \cdots + a_{r-1}\rho(n-r+1,r), \tag{1.3}$$

for each $n \geq r$, with $\rho(r,r) = 1$ and $\rho(n,r) = 0$ for $0 \leq n \leq r-1$. Therefore, we have that the sequence $\{\rho(n+1,r)\}_{n\geq 0}$ satisfies the Equation (1.1) with initial conditions $\rho(1,r) = \cdots = \rho(r-1,r) = 0$ and $\rho(r,r) = 1$. Consider the family of the sequences $\{\rho(n+1,s)\}_{n\geq 0}$, for $0 \leq s \leq r-1$ denoted by

$$\mathbb{L} = \{\{\rho(n+1,s)\}_{n\geq 0}, 1 \leq s \leq r\}.$$

where each sequence $\{\rho(n+1,s)\}_{n\geq 0}$ is given by

$$\begin{aligned} \rho(n+1,s) &= \sum_{i=0}^{r-1} a_i\rho(n-i,s) \qquad \text{for} \qquad n \geq r, \\ \rho(s,s) &= 1 \text{ and } \rho(n,s) = 0 \text{ for } 1 \leq n \neq s \leq r-1. \end{aligned} \tag{1.4}$$

In the present study, we focus on explicitly describing the closed connection between the family $\mathbb{L}$ and the fundamental sequence of numbers $\{\rho(n+1,r)\}_{n\geq 0}$, providing combinatorial identities. This approach is another perspective of has been done in the literature, especially, those of [4], [15] and [16]. Finally, significant illustrative examples and applications are furnished and some perspectives are proposed. Consequently, we get various generalized identities. Moreover, some results of the literature are recovered.

2 The matrix formulation and the family $\mathbb{L}$

In this section, we will talk about the matrix form associated with the family $\mathbb{L}$, highlighting its properties. In addition, we will introduce some new interesting identities.

2.1 The matrix formulation and properties

Consider the matrix

$$\mathbb{M}_n = \begin{pmatrix} \rho(n+r,r) & \cdots & \rho(n+r,j) & \cdots & \rho(n+r,1) \\ \vdots & \vdots & \vdots & & \vdots \\ \rho(n+1,r) & \cdots & \rho(n+1,j) & \cdots & \rho(n+1,1) \end{pmatrix}. \tag{2.1}$$

The next Lemma was introduced in [Proposition 2.1,[6]] and it deals with the relationship between the matrix $\mathbb{M}_n$ and the companion matrix associated with the family $\mathbb{L}$.

Lemma 2.1. *(See [1, 6]) For each $n \geq 0$, we have that $\mathbb{A}^n = \mathbb{M}_n$, where $\mathbb{A}$ is the classical companion matrix:*

$$\mathbb{A} = \begin{bmatrix} a_0 & a_1 & a_2 & \dots & a_{r-1} \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix},$$

or yet, $a_{i,j}^{(n)} = \rho(n+r-i-1, r-j+1)$.

Since determinant of $\mathbb{A}^n$ is equal to $a_{r-1}(-1)^{n(r-1)}$, and $a_{r-1} \neq 0$, then determinant of $\mathbb{M}_n \neq 0$. Moreover, if we take any sequence v_n of type (1.1) with initial conditions (1.2), a direct computation gives us that

$$v_n = \alpha_0\rho(n+1,1) + \alpha_1\rho(n+1,2) + \cdots + \alpha_{r-1}\rho(n+1,r). \tag{2.2}$$

Therefore, the family $\mathbb{L}$ is a base of the $\mathbb{K}$-vector space of finite dimension r, of solutions of Equation (1.1).

2.2 Some Properties

In this subsection, we will describe some interesting properties of the family $\mathbb{L}$. More specifically, we will describe the element $\rho(n+1,j)$, for $1 \leq j \leq r-1$ in terms of the elements of the sequence $\{\rho(n+1,r)\}_{n\geq 0.}$ The proof of the following result is done by induction on n and will be omitted.

Proposition 2.2. Consider the family $\mathbb{L} = \{\{\rho(n+1,s)\}_{n\geq 0}, 1 \leq s \leq r\}$. Then we have

$$\rho(n+1,1) = a_{r-1}\rho(n,r), \tag{2.3}$$

for $n \geq 1$, and for each j, with $2 \leq j \leq r-1$, holds

$$\rho(n+1,j) = \sum_{i=0}^{j-1} a_{r-j+i}\rho(n-i,r), \tag{2.4}$$

for $n \geq j$.

Combining identities (2.2), (2.3) and (2.4), we have the following general result.

Proposition 2.3. Let $\{w_n\}_{n\geq 0}$ be a $r-$generalized Fibonacci sequence of type (1.1), with initial conditions $\alpha_0, \alpha_1, \cdots, \alpha_{r-1}$. Then, for all $n \geq 0$, holds:

$$w_n = \alpha_0 a_{r-1}\rho(n,r) + \cdots + \alpha_{j-1}\sum_{i=0}^{j-1} a_{r-j+i}\rho(n-i,r) + \cdots + \alpha_{r-1}\rho(n+1,r). \tag{2.5}$$

Proposition 2.3 gives us a general combinatorial expression for any sequence of type (1.1), with arbitrary initial conditions $\alpha_0, \alpha_1, \cdots, \alpha_{r-1}$.

2.3 Combinatorial Identities

Let $\widehat{C}(n)$ be the Casoratian matrix associated to the family $\mathbb{L}$, defined by

$$\widehat{C}(n) = \begin{pmatrix} \rho(n+1,r) & \cdots & \rho(n+1,j) & \rho(n+1,r) \\ \vdots & \cdots & \vdots & \vdots \\ \rho(n+r,r) & \cdots & \rho(n+r,j) & \rho(n+r,1) \end{pmatrix}. \tag{2.6}$$

We have $\widehat{C}(n) = J \times \mathbb{M}_n \times J$, where $J = (b_{i,j})_{1\leq i,j\leq r}$ is the anti-diagonal unit matrix, whose entries are given by $b_{i,j} = 1$ for $i + j = r + 1$ and $b_{i,j} = 0$ otherwise. Therefore $\hat{C}(n+m) = J\mathbb{A}^{n+m}J = J\mathbb{A}^n\mathbb{A}^m J$. Since for every positive integers n and m is verified Casoratian matrix property, $\hat{C}(n+m) = \hat{C}(n)\hat{C}(m)$, then $\mathbb{A}^{m+n} = \mathbb{A}^m\mathbb{A}^n = \mathbb{A}^n\mathbb{A}^m$.

Next, we obtain a result by using the Casoratian matrix property.

Proposition 2.4. Consider the family $\mathbb{L} = \{\{\rho(n+1,s)\}_{n\geq 0}, 1 \leq s \leq r\}$. Then, for each $m, n \geq 0$, following identity is verified,

$$\rho(m+n+r-i+1, r-j+1) = \sum_{k=1}^{r} \rho(n+r-i+1, r-k+1)\rho(m+r-k+1, r-j+1). \tag{2.7}$$

Proof. If we denote $\mathbb{A}^{m+n} = (a_{ij}^{(m+n)})_{1\leq i,j\leq r}$, it follows,

$$a_{ij}^{(m+n)} = \sum_{k=1}^{r} a_{ik}^{(m)} a_{kj}^{(n)} = \sum_{k=1}^{r} a_{ik}^{(n)} a_{kj}^{(m)}. \tag{2.8}$$

But by Lemma 2.1 the identity $a_{ij}^{(n)} = \rho(n+r-i+1, r-j+1)$ is verified. Thus, $a_{ik}^{(n)} = \rho(n+r-i+1, r-k+1)$ and $a_{kj}^{(m)} = \rho(m+r-k+1, r-j+1)$.

Hence, we have

$$a_{ij}^{(m+n)} = \sum_{k=1}^{r} \rho(n+r-i+1, r-k+1)\rho(m+r-k+1, r-j+1). \tag{2.9}$$

Since $a_{ij}^{(m+n)} = \rho(m+n+r-i+1, r-j+1)$, then it is verified

$$\rho(m+n+r-i+1, r-j+1) = \sum_{k=1}^{r} \rho(n+r-i+1, r-k+1)\rho(m+r-k+1, r-j+1).$$

□

By using identities (2.2), (2.3) and (2.4), we obtain the Theorem below.

Theorem 2.5. *Consider the family* $\mathbb{L} = \{\{\rho(n+1, s)\}_{n\geq 0}, 1 \leq s \leq r\}$. *Then for every* $m, n \geq 0$, *it is verified the following identities:*

$$\rho(m+n+p, q) = \sum_{d=1}^{r} \left[\sum_{i=0}^{d-1} a_{r-d+i}\rho(m+p-i, r)\right] \left[\sum_{j=0}^{q-1} a_{r-q+j}\rho(n+d-1-j, r)\right], \tag{2.10}$$

and

$$\rho(m+s+1, r) = \sum_{d=1}^{r}\sum_{i=1}^{d} a_{r-d+i}\rho(m-i, r)\rho(s+d, r). \tag{2.11}$$

Observe that, Casoratian matrix property can be extended, i.e, for every positive integers $m_1, m_2, \ldots, m_t$, it is verified

$$\hat{C}(m_1 + m_2 + \ldots + m_t) = \hat{C}(m_1)\hat{C}(m_2)\cdots\hat{C}(m_t),$$

then $\mathbb{A}^{m_1+m_2+\ldots+m_t} = \mathbb{A}^{m_1}.\mathbb{A}^{m_2}\cdots.\mathbb{A}^{m_t}$. Therefore, similarly as did previously, we can obtain identities involving sums and products of elements of $\mathbb{L}$, as follows for $t = 3$, below.

Proposition 2.6. Consider the family $\mathbb{L} = \{\{\rho(n+1,s)\}_{n\geq 0}, 1 \leq s \leq r\}$. Then, for all positive integers m_1, m_2 and m_3, holds:

$$\rho(m_1+m_2+m_3+r-i+1, r-j+1) = \sum_{k=1}^{r}\left(\sum_{l=1}^{r}\rho(m_1+r-i+1, r-l+1)\rho(m_2+r-l+1, r-k+1)\right) \times\rho(m_3+r-k+1, r-j+1). \tag{2.12}$$

Theorem 2.7. *Consider the family $\mathbb{L} = \{\{\rho(n+1,s)\}_{n\geq 0}, 1 \leq s \leq r\}$. Then for every for all positive integers m_1, m_2 and m_3, holds:*

$$\rho(m_1+m_2+m_3+1, r) = \sum_{k=1}^{r}\left(\sum_{l=1}^{r}\rho(m_1+1, r-l+1)\rho(m_2+r-l+1, r-k+1)\right) \times\rho(m_3+r-k+1, r). \tag{2.13}$$

As a particular case, by evaluating $r = 2$, Equation (2.12) and (2.7) assumes, respectively, the following forms below.

Corollary 2.8. The following combinatorial identities are verified,

$$\rho(m+n+1,2) = a_2[\rho(n,2)\rho(n+1,2)+\rho(m-1,2)\rho(n+2,2)]+a_1\rho(m,2)\rho(n+2,2), \tag{2.14}$$

for each $m \geq 1$ and $n \geq 0$, and

$$\begin{aligned}\rho(m_1+m_2+m_3+1,2) &= \\ = (\rho(m_1+1,2)\rho(m_2+2,2) &+ a_2\rho(m_1,2)\rho(m_2+1,2)) \\ &\times\rho(m_3+2,2) \\ + \big(\rho(m_1+1,2)a_2\rho(m_2+1,2) &+ (a_2)^2\rho(m_1,2)\rho(m_2,2)\big) \\ &\times\rho(m_3+1,2)\end{aligned} \tag{2.15}$$

for each $m_1 \geq 2$, $m_2 \geq 1$ and $m_3 \geq 0$.

3 Application: Special Cases

In this section, we will show applications of the results obtained in the Fibonacci $r-$generalized fundamental system to some special cases: The Generalized Fibonacci numbers, The Generalized Pell numbers, The Generalized Jacobsthal numbers, and The Model of generalized Pell numbers. For more details, the reader can refer to the following articles [15], [16], [9], [8] and [4].

3.1 The Generalized Fibonacci numbers

Consider the coefficients $a_0 = a_1 = \cdots = a_{r-1} = 1$ in the Equation (1.1), then we get the generalized Fibonacci numbers given by,

$$F_n = \sum_{i=0}^{r-1} F_{n-i-1} \qquad \text{for} \qquad n \geq r, \tag{3.1}$$

with initial condition $F_n = \alpha_n$ for $n = 0,\ 1,\ ...,\ r-1$. Family $\mathbb{L}$ associated to F_n is given by $\{\{\rho_F(n+1,s)\}_{n\geq 0}, 1 \leq s \leq r\}$, where

$$\rho_F(n+1,r) = \sum_{k_0+2k_1+\cdots+rk_{r-1}=n-r+1} \frac{(k_0+\cdots+k_{r-1})!}{k_0!k_1!\cdots k_{r-1}!}, n \geq r.$$

Equation (2.5) allow us obtain the combinatorial expression for F_n in terms of the elements of the fundamental sequence $\rho_F(n+1,r)$, given by

$$F_n = \alpha_0\rho_F(n,r) + \cdots + \alpha_{j-1}\sum_{i=0}^{j-1}\rho_F(n-i,r) + \cdots + \alpha_{r-1}\rho_F(n+1,r). \tag{3.2}$$

Following the Expressions (2.11) and (3.2) we obtain the following result for generalized Fibonacci numbers.

Proposition 3.1. Consider the family $\mathbb{L} = \{\{\rho(n+1,s)\}_{n\geq 0}, 1 \leq s \leq r\}$. Then, for every $n, m \geq 0$, holds

$$\begin{aligned} F_{m+n} = \alpha_0 \sum_{d=1}^{r}\sum_{k=1}^{d} \rho_F(m-k,r)\rho_F(n-1+d,r) + \cdots \\ +\alpha_{j-1}\sum_{i=0}^{j-1}\sum_{d=1}^{r}\sum_{k=1}^{d} \rho_F(m-k,r)\rho_F(n-i+d,r) \\ + \cdots + \alpha_{r-1}\sum_{d=1}^{r}\sum_{k=1}^{d} \rho_F(m-k,r)\rho_F(n+d,r), \end{aligned}$$

where F_n is the n-th generalized Fibonacci number.

3.2 The Generalized Pell numbers

By replacing $a_0 = 2$ and $a_1 = \cdots = a_{r-1} = 1$ in the Equation (1.1), we get the generalized Pell numbers given by,

$$P_n = 2P_{n-1} + \sum_{i=1}^{r-1} P_{n-i-1} \qquad \text{for} \qquad n \geq r, \tag{3.3}$$

with initial condition $P_n = \alpha_n$ for $n = 0,\ 1,\ ...,\ r-1$. Family $\mathbb{L}$ associated to P_n is given by $\{\{\rho_P(n+1,s)\}_{n\geq 0}, 1 \leq s \leq r\}$, where

$$\rho_P(n+1,r) = \sum_{k_0+2k_1+\cdots+rk_{r-1}=n-r+1} \frac{(k_0+\cdots+k_{r-1})!}{k_0!k_1!\cdots k_{r-1}!} 2^{k_0}, n \geq r.$$

Equation (2.5) allow us obtain the combinatorial expression for P_n given as follows:

$$P_n = \alpha_0 a_{r-1}\rho_P(n,r) + \cdots + \alpha_{j-1}\sum_{i=0}^{j-1} a_{r-j+i}\rho_P(n-i,r) + \cdots + \alpha_{r-1}\rho_P(n+1,r). \tag{3.4}$$

Moreover, we have the following result for the generalized Pell numbers.

Proposition 3.2. Consider the family $\mathbb{L} = \{\{\rho(n+1,s)\}_{n\geq 0}, 1 \leq s \leq r\}$. Then, for every $n, m \geq 0$, holds

$$\begin{aligned} P_{m+n} = \alpha_0 &\sum_{d=1}^{r}\sum_{k=1}^{d} \rho_P(m-k,r)\rho_P(n-1+d,r) + \cdots \\ &+\alpha_{j-1}\sum_{i=0}^{j-1}\sum_{d=1}^{r}\sum_{k=1}^{d} \rho_P(m-k,r)(n-i+d,r) \\ &+ \cdots + \alpha_{r-1}\sum_{d=1}^{r}\sum_{k=1}^{d} \rho_P(m-k,r)\rho_P(n+d,r), \end{aligned}$$

where P_n is the n-th generalized Pell number.

By using the classical initial conditions $\alpha_j = 0$, for $0 \leq j \leq r-2$ and $\alpha_{r-1} = 1$, we obtain the particular case

$$P_{m+s} = \sum_{d=1}^{r}\sum_{i=1}^{d} \rho_P(m-i,r)\rho_P(s+d,r). \tag{3.5}$$

Expression (3.5) was established in [Section 3.2,[16]] . Then, it seems to us that Proposition 3.2 is a generalization of an identity presented in [16].

3.3 The Generalized Jacobsthal numbers

Consider the coefficients $a_0 = a_1 = \cdots = a_{r-2} = 1$ and $a_{r-1} = 2$ in the Equation (1.1), then we get the generalized Jacobsthal numbers given by,

$$J_n = \sum_{i=0}^{r-2} J_{n-i-1} + 2J_{n-r} \qquad \text{for} \qquad n \geq r, \tag{3.6}$$

with initial condition $J_n = \alpha_n$ for $n = 0,\ 1,\ ...,\ r-1$. Therefore, by Equation (2.5), the combinatorial expression for J_n is given by

$$J_n = 2\alpha_0\rho_J(n,r) + \cdots + \alpha_{j-1}\sum_{i=0}^{j-1} a_{r-j+i}\rho_J(n-i,r) + \cdots + \alpha_{r-1}\rho_J(n+1,r), \tag{3.7}$$

where

$$\rho_J(n+1,r) = \sum_{k_0+2k_1+\cdots+rk_{r-1}=n-r+1} \frac{(k_0+\cdots+k_{r-1})!}{k_0!k_1!\cdots k_{r-1}!}2^{k_{r-1}}, n \geq r.$$

In addition, we have the following proposition.

Proposition 3.3. Consider the family $\mathbb{L} = \{\{\rho_J(n+1,s)\}_{n\geq 0}, 1 \leq s \leq r\}$. Then, for every $n, m \geq 0$, holds

$$\begin{aligned} P_{m+n} = 2\alpha_0 &\sum_{d=1}^{r}\sum_{k=1}^{d} \rho_P(m-k,r)\rho_P(n-1+d,r) + \cdots \\ +\alpha_{j-1}\sum_{i=0}^{j-1} a_{r-j+i} &\sum_{d=1}^{r}\sum_{k=1}^{d} \rho_P(m-k,r)\rho_P(n-i+d,r) \\ &+ \cdots + \alpha_{r-1}\sum_{d=1}^{r}\sum_{k=1}^{d} \rho_P(m-k,r)\rho_P(n+d,r), \end{aligned}$$

where J_n is the n-th generalized Jacobsthal number.

Corollary 3.4. Let $\{J_n\}_{n\geq 0}$ be the generalized Jacobsthal numbers with initial conditions $\alpha_j = 0$, for $0 \leq j \leq r-2$ and $\alpha_{r-1} = 1$. Then, for every $n, m \geq 0$, we have the following identity:

$$J_{m+n} = \sum_{d=1}^{r}\left[\sum_{i=1}^{d-1} \rho_J(m-i+1,r) + 2\rho_J(m-d+1,r)\right]\rho_J(n+d,r). \tag{3.8}$$

3.4 The Model of generalized Pell numbers

Consider the coefficients given by $a_0 = 2^d$, $a_1 = \cdots = a_i = 0$; $a_{i+1} = \cdots = a_{r-2} = 1$ and $a_{r-1} = h \neq 0$ in the Equation (1.1). We get the model of generalized Pell numbers,

$$P_{i,n+1} = 2^d P_{i,n} + \sum_{k=i+1}^{r-2} P_{i,n-k} + hP_{i,n-r+1} \qquad \text{for} \qquad n \geq r-1, \tag{3.9}$$

The combinatorial expression for $P_{i,n+1}$ is given by

$$P_{i,n+1} = \alpha_0 h \rho_{P_i}(n,r) + \cdots + \alpha_{j-1} \sum_{i=0}^{j-1} a_{r-j+i} \rho_{P_i}(n-i,r) + \cdots + \alpha_{r-1} \rho_{P_i}(n+1,r), \tag{3.10}$$

where

$$\rho_{P_i}(n+1,r) = \sum_{k_0+2k_1+\cdots+rk_{r-1}=n-r+1} \frac{(k_0+\cdots+k_{r-1})!}{k_0!k_1!\cdots k_{r-1}!} 2^{dk_0} h^{k_{r-1}}, n \geq r. \tag{3.11}$$

Next, we have a new Proposition.

Proposition 3.5. Consider the family $\mathbb{L} = \{\{\rho_{P_i}(n+1,s)\}_{n\geq 0}, 1 \leq s \leq r\}$. Then, for every $n, m \geq 0$, holds

$$\begin{aligned} P_{i,n+m} = {} & \alpha_0 h \sum_{d=1}^{r} \sum_{k=1}^{d} \rho_{P_i}(m-k,r)\rho_{P_i}(n-1+d,r) + \\ & + \alpha_i \sum_{t=0}^{i} a_{r-t+i} \sum_{d=1}^{r} \sum_{k=1}^{d} \rho_{P_i}(m-k,r)\rho_{P_i}(n-i+d,r) \\ & + \cdots + \alpha_{r-1} \sum_{d=1}^{r} \sum_{k=1}^{d} \rho_{P_i}(m-k,r)\rho_{P_i}(n+d,r), \end{aligned}$$

where $P_{i,n}$ is given by (3.9).

Corollary 3.6. Let $\{P_{i,n}\}_{n\geq 0}$ be the model of generalized Pell numbers with initial conditions $\alpha_j = 0$, for $0 \leq j \leq r-2$ and $\alpha_{r-1} = 1$. Then, for every $s, m \geq 0$, we have the following identity:

$$P_{i,m+s} = \sum_{d=1}^{r} \left[\sum_{j=d-r+i+1}^{d-2} \Delta_{m,s} \right] \rho_i(s+d,r), \tag{3.12}$$

where $\Delta_{m,s} = \rho_i(m-j,r) + h\rho_i(m-s+1,r)$.

4 Concluding remarks and perspective

In this study, we established a combinatorial fundamental system related to the sequences of $r-$generalized Fibonacci type, called family $\mathbb{L}$. Moreover, by using the properties of the Casoratian matrix associated with the family $\mathbb{L}$ we obtain new combinatorial identities and properties. In particular cases, we studied the generalized Fibonacci, Pell, Jacobsthal, and model of Pell numbers, and provided new combinatorial identities for these classes of numbers. It seems to us that the approach presented here is new in the literature, and the subject can be provided from other perspectives.

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