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A NOTE ON BELL POLYNOMIALS AND COLORED PARTITIONS

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Abstract

In this short communication we will explore Bell's exponential function in order to obtain new identities for a series involving the function $p_k(n)$, the number of partitions of n into k colors.

Keywords: $k-$colored partitions, Convergent series, Bell polynomials.

1 Introduction and Background

In this article we will find convergent series that relate the function that counts the number of colored partitions with the exponential function using Bell polynomials. Bell polynomials, named after Eric Temple Bell, are a family of polynomials significant in combinatorics, particularly in the study of partitions of sets, moments of probability distributions, and in the theory of differential equations. There are two primary types of Bell polynomials: the *exponential Bell polynomials* and the *complete (or partial) Bell polynomials*, (see, for instance, [3, 4]).

The partial Bell polynomial and complete exponential Bell partition polynomial, as defined in Chapter 11 of Charalambides [5], [8] and [9], are given respectively by the sums:

$$B_{n,j}(x_1, x_2, \dots, x_{n-j+1})$$

$$= \sum_{\substack{k_1+2k_2+\ldots+nk_n=n \\ k_i\geq 0 \\ k_1+k_2+\ldots+k_{n-j+1}=j}} \frac{n!}{k_1(1!)^{k_1}k_2!(2!)^{k_2}\cdots k_{n-j+1}!((n-j+1)!)^{k_{n-j+1}}} x_1^{k_1}x_2^{k_2}\cdots x_{n-j+1}^{k_{n-j+1}}, \tag{1.1}$$

$$B_n(x_1,x_2,\ldots,x_n) = \sum_{\substack{k_1+2k_2+\ldots+(n-j+1)k_{n-j+1}=n \\ k_i\geq 0}} \frac{n!}{k_1(1!)^{k_1}k_2!(2!)^{k_2}\cdots k_n!(n!)^{k_n}} x_1^{k_1}x_2^{k_2}\cdots x_n^{k_n}. \tag{1.2}$$

The generating function for the exponential Bell partition polynomial (1.2) is

$$\sum_{n=0}^{\infty} B_n(x_1,\ldots,x_n)\frac{t^n}{n!} = \exp\left(\sum_{j=1}^{\infty} x_j \frac{t^j}{j!}\right).$$

In addition, these are two important and well-known properties about Bell polynomials:

$$B_{n,j}(bx_1, bx^2\ldots, bx_n)\frac{t^n}{n!} = b^j B_{n,j}(x_1,x_2,\ldots,x_n). \tag{1.3}$$

$$B_n(x_1,x_2,\ldots,x_n) = \sum_{j=0}^{n} B_{n,j}(x_1,x_2,\ldots,x_n). \tag{1.4}$$

The exponential Bell polynomial encodes the information related to the ways a set can be partitioned. Thus, the number of monomials that appear in the partial Bell polynomial is equal to the number of ways the integer n can be expressed as a summation of k positive integers. More precisely, Bell polynomial describe the number of partitions of n.

A partition[1] of an integer n is an unordered collection of integers $(\lambda_1,\lambda_2,\ldots,\lambda_s)$ such that $\lambda_1+\lambda_2+\ldots+\lambda_s = n$. We agree that $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_s$, and each λ_i is called a part of the partition. A k-colored partition of n is an integer partition of n in which each part receives one color of k available colors. We denote by $p_k(n)$ the number of k-colored integer partitions of n. For example $p_2(4) = 20$, and these partitions are: 4, 4, $3+1$, $3+1$, $3+1$, $3+1$, $2+2$, $2+2$, $2+2$, $2+1+1$, $2+1+1$, $2+1+1$, $2+1+1$, $2+1+1$, $2+1+1$, $1+1+1+1$, $1+1+1+1$, $1+1+1+1$, $1+1+1+1$ and $1+1+1+1$.

The generating function for the sequence $(p_k(n))$ is given by the next infinite product.

[1]See Andrews [1] and Chern et al [6] for more detailed information.

$$\sum_{n=0}^{\infty} p_k(n)q^n = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^k},$$

where $p_k(0) = 1$. For simplicity we denote $p_1(n) = p(n)$.

A composition of an integer n, as defined in Heubach and Mansour in [10], is a partition in which the order of the parts matters. For example the compositions of $n = 3$ are: $(1, 1, 1)$, $(2, 1)$, $(1, 2)$ and 3. We denote the set of compositions of n by C_n. The number of compositions of n is 2^{n-1}.

The biggest inspiration for our main result are the following theorems by Alegri and Spreafico [2].

Theorem 1.1 (Theorem 2 of [2]). *For a positive integer n, the following identity is valid*

$$\sum_{k=0}^{n-1} p_1(k) \sum_{w_1+\ldots+w_m \in C(n-k)} \sum_{1\le l_1<\ldots<l_m} \frac{(-1)^m p_2(w_1)\cdots p_2(w_m)}{l_1^2\cdots l_m^2\pi^{2m}} \prod_{l\neq l_1,\ldots,l_m} \left(1-\frac{1}{l^2\pi^2}\right) \quad (1.5)$$

$$+p_1(n)sin(1) = \sum_{k=0}^{\infty} \frac{(-1)^k p_{2k+1}(n)}{(2k+1)!}.$$

Theorem 1.2 (Theorem 3 of [2]). *For a positive integer n, under the previous notations, the following identity is valid*

$$\sum_{w_1+\ldots+w_m \in C(n)} \sum_{1\le l_1<\ldots<l_m} \frac{(-1)^m 2^{2m} p_2(w_1)\cdots p_2(w_m)}{(2l_1-1)^2\cdots(2l_m-1)^2\pi^{2m}} \prod_{l\neq l_1,\ldots,l_m} \left(1-\frac{4}{(2l-1)^2\pi^2}\right) \quad (1.6)$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k p_{2k}(n)}{(2k)!}$$

In these theorems the authors had explore the functions sine, cosine and the generating function for the k-colored integer partitions of n. In these paper, we will provide an explicit exponential formula for the k-colored integer partitions of n in terms of the partial Bell polynomials. This connection provide a new combinatorial interpretation for partial Bell polynomial, as well as, new identities involving partition functions.

The next section is devoted to establish these new identities.

2 Identities

In order to obtain our results, we must consider the generating function for the number of k-colored partitions of n given by

$$\sum_{n=0}^{\infty} n!p_k(n)\frac{q^n}{n!} = \left(\sum_{n=0}^{\infty} n!p(n)\frac{q^n}{n!}\right)^k. \tag{2.1}$$

By Shattuck [11] (Theorem 7, and Remark) and Comtet [7] (page 141) we have the potential function $p_k(n) = p_k(n)(1!p(1), 2!p(2), \ldots)$ and

$$n!p_k(n) = \sum_{j=0}^{n} \frac{k!}{(k-j)!} B_{n,j}(1!p(1), \ldots, (n-j+1)!p(n-j+1)). \tag{2.2}$$

Therefore, by expressions (2.1) and (2.2), we obtain

$$\begin{aligned}\sum_{k=0}^{\infty} \frac{(-x)^k}{k!} p_k(n) &= \frac{1}{n!}\sum_{k=0}^{\infty} \frac{(-x)^k}{k!} \sum_{j=0}^{n} \frac{k!}{(k-j)!} B_{n,j}(1!p(1), \ldots, (n-j+1)!p(n-j+1)) \\ &= \frac{1}{n!}\sum_{j=0}^{n}\sum_{k=0}^{\infty} \frac{(-x)^j}{(k-j)!} B_{n,j}(1!p(1), \ldots, (n-j+1)!p(n-j+1)) \\ &= \frac{1}{n!}\sum_{j=0}^{n} B_{n,j}(1!p(1), \ldots, (n-j+1)!p(n-j+1))Q(j),\end{aligned}$$

where $Q(j) = \sum_{k=j}^{\infty} \frac{(-x)^k}{(k-j)!} = e^{-x}(-x)^j$. Thus,

$$\sum_{k=0}^{\infty} \frac{(-x)^k}{k!} p_k(n) = \frac{e^{-x}}{n!}\sum_{j=0}^{n} B_{n,j}(-1!xp(1), -2!xp(2), \ldots, -(n-j+1)!xp(n-j+1)).$$

By identities (1.3) and (1.4), we get

$$\sum_{k=0}^{\infty} \frac{(-x)^k}{k!} p_k(n) = \frac{e^{-x}}{n!} B_n(-1!xp(1), -2!xp(2), \ldots, -(n)!xp(n)). \tag{2.3}$$

Therefore, by identities (1.2) and (2.3), we establish the following result, that is the core of this article.

Theorem 2.1.

$$\sum_{k=0}^{\infty} \frac{(-x)^k}{(k)!} p_k(n) = e^{-x} \sum_{\substack{k_1+2k_2+\ldots+nk_n=n \\ k_i\geq 0}} \frac{(-x)^{k_1+k_2+\cdots+k_n}(p(1))^{k_1}(p(2))^{k_2}\cdots(p(n))^{k_n}}{k_1!k_2!\cdots k_n!}. \tag{2.4}$$

Now, by evaluating $x = -\theta i$ in (2.4), for a real number θ, we get the following result.

Corollary 2.2. The following identity holds

$$\sum_{k=0}^{\infty} \frac{(-1)^k p_{2k}(n)}{(2k)!}\theta^{2k} + i\sum_{k=0}^{\infty} \frac{(-1)^k p_{2k+1}(n)}{(2k+1)!}\theta^{2k+1}$$
$$= e^{\theta i} \sum_{\substack{k_1+2k_2+\ldots+nk_n=n \\ k_i\geq 0}} \frac{(\theta i)^{k_1+k_2+\cdots+k_n}(p(1))^{k_1}(p(2))^{k_2}\cdots(p(n))^{k_n}}{k_1!k_2!\cdots k_n!}.$$

In addition, for $x = -i$ in (2.4) and by combining the results of Theorems 1.1 and 1.2, we have the next identity.

Corollary 2.3.

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}p_{2k}(n) - i\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}p_{2k+1}(n) = e^{-i} \sum_{\substack{k_1+2k_2+\ldots+nk_n=n \\ k_i\geq 0}} \frac{(p(1))^{k_1}(p(2))^{k_2}\cdots(p(n))^{k_n}}{k_1!k_2!\cdots k_n!}$$

$$= \sum_{w_1+\ldots+w_m\in C(n)} \sum_{1\leq l_1<\ldots<l_m} \frac{(-1)^m 2^{2m} p_2(w_1)\cdots p_2(w_m)}{(2l_1-1)^2\cdots(2l_m-1)^2\pi^{2m}} \prod_{l\neq l_1,\ldots,l_m}\left(1-\frac{4}{(2l-1)^2\pi^2}\right)$$

$$-i\left(\sum_{k=0}^{n-1} p_1(k) \sum_{w_1+\ldots+w_m\in C(n-k)} \sum_{1\leq l_1<\ldots<l_m} \frac{(-1)^m p_2(w_1)\cdots p_2(w_m)}{l_1^2\cdots l_m^2\pi^{2m}} \prod_{l\neq l_1,\ldots,l_m}\left(1-\frac{1}{l^2\pi^2}\right) + p_1(n)sin(1)\right)$$

Now, by splitting equation (2.4) into real and imaginary parts, we have the next identities.

Corollary 2.4.

$$\sum_{k=0}^{\infty} \frac{(-1)^k p_{2k}(n)}{(2k)!}\theta^{2k} = \sum_{\substack{k_1+2k_2+\ldots+nk_n=n \\ k_1+k_2+\ldots+k_n \ even}} \frac{(\theta i)^{k_1+k_2+\cdots+k_n}(p(1))^{k_1}(p(2))^{k_2}\cdots(p(n))^{k_n}cos\theta}{k_1!k_2!\cdots k_n!}$$

$$+i\left(\sum_{\substack{k_1+2k_2+\ldots+nk_n=n \\ k_1+k_2+\ldots+k_n \ odd}} \frac{(\theta i)^{k_1+k_2+\cdots+k_n}(p(1))^{k_1}(p(2))^{k_2}\cdots(p(n))^{k_n}sin\theta}{k_1!k_2!\cdots k_n!}\right)$$

$$\sum_{k=0}^{\infty}\frac{(-1)^k p_{2k+1}(n)}{(2k+1)!}\theta^{2k+1} = -i\left(\sum_{\substack{k_1+2k_2+\ldots+nk_n=n\\ k_1+k_2+\ldots+k_n\ odd}}\frac{(\theta i)^{k_1+k_2+\cdots+k_n}(p(1))^{k_1}(p(2))^{k_2}\cdots(p(n))^{k_n}cos\theta}{k_1!k_2!\cdots k_n!}\right.$$
$$\left.+\sum_{\substack{k_1+2k_2+\ldots+nk_n=n\\ k_1+k_2+\ldots+k_n\ even}}\frac{(\theta i)^{k_1+k_2+\cdots+k_n}(p(1))^{k_1}(p(2))^{k_2}\cdots(p(n))^{k_n}sin\theta}{k_1!k_2!\cdots k_n!}\right)$$

3 Conclusion

In this paper we established a combinatorial interpretation of Bell's exponential function is terms of the number of partitions of n into k colors. For do this, we provided an exponential formula for the k-colored partition function in terms of the Bell's exponential function, and derived some identities by evaluating the function for functions sine,cosine in a real and imaginary numbers.

It seems to us that this combinatorial approach is new in literature, and and it is a subject that can still be explored further.

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