REVISEM Revista Sergipana de Matemática e Educação Matemática

https://doi.org/10.34179/revisem.v9i3.21793

Special Issue in honor of the 73rd birthday of Professor José Plínio de Oliveira Santos (IMECC-UNICAMP)

ARITHMETIC FUNCTIONS VERIFYING A RECURRENCE RELATION, COMPOSITIONS, AND BELL POLYNOMIALS

Juan D. Bulnes
Universidade Federal do Amapá, Brazil.
bulnes@unifap.br
Taekyun Kim
Kwangwoon University, Republic of Korea.
tkkim@kw.ac.kr
José L. López-Bonilla
Instituto Politécnico Nacional, México.
jlopezb@ipn.mx
Mateus Alegri
Universidade Federal de Sergipe, Brazil.
malegri@academico.ufs.br

Abstract

In the work we apply the Z-transform to the recurrence of Cauchy convolution type, satisfied by several arithmetic functions, to obtain its solution in terms of the complete Bell polynomials. One of the most important arithmetic function used here is $\sigma_1(n)$, the function that sum all positive divisors of n. Our main result can be applied to find a closed formula for the number of k-colored partitions, sum of triangular numbers and more.

Keywords: $Z-$transforms, $k-$colored partitions, Arithmetic functions, Recurrence relations, Compositions, Bell polynomials.

1 Introduction and Background

In this paper, our basic aim is to obtain recurrence relations with the structure of a Cauchy convolution[1] as follows.

$$nf_k(n) = k\sum_{j=1}^{n} g(j)f_k(n-j), \quad k \geq 1, \quad n \geq 0 \tag{1.1}$$

[1]More detailed information about the Cauchy convolution can be found in [44].

where $f_k(0) = 1\ \forall\ k$ and $g(0) = 0$. In Section 2 we show that the $Z-$transform as used in [11, 9, 14], allows to obtain the following solution of the recurrence (1.1) in terms of the complete Bell polynomials (as defined in [13, 12, 8, 16, 17, 18, 41, 42]):

$$f_k(n) = \frac{1}{n!}B_n\Big(0!kg(1), 1!kg(2), 2!kg(3), \ldots, (n-1)!kg(n)\Big). \tag{1.2}$$

In order to do this we recall some concepts. Still in Section 2, we show some results associated to colored integer partitions and compositions[2]. A partition of an integer n is an unordered collection of integers $(\lambda_1, \lambda_2, \ldots, \lambda_s)$ such that $\lambda_1 + \lambda_2 + \ldots + \lambda_s = n$. We agree that $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_s$, and each λ_i is called a part of the partition. A $k-$colored partition of n is an integer partition of n in which each part receives one color of k available colors. We denote by $p_k(n)$ the number of $k-$ colored integer partitions of n. For example $p_2(4) = 20$, and these partitions are: 4, 4, $3+1$, $3+1$, $3+1$, $3+1$, $2+2$, $2+2$, $2+2$, $2+1+1$, $2+1+1$, $2+1+1$, $2+1+1$, $2+1+1$, $2+1+1$, $1+1+1+1$, $1+1+1+1$, $1+1+1+1$, $1+1+1+1$ and $1+1+1+1$.

The generating function for the sequence $(p_k(n))$ is given by the next infinite product.

$$\sum_{n=0}^{\infty} p_k(n)q^n = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^k},$$

where $p_k(0) = 1$.

The generating function for the number of partitions of n whose parts are distinct is given by:

$$\sum_{n=0}^{\infty} p_D(n)q^n = \prod_{n=1}^{\infty}(1+q^n),$$

where $p_D(n)$ denotes the number of this class of partitions for a positive integer n.

A composition of an integer n is a partition in which the order of the parts matters. For example the compositions of $n = 3$ are: $(1,1,1)$, $(2,1)$, $(1,2)$ and 3. We denote the set of compositions of n by C_n. The number of compositions of n is 2^{n-1}.

The exponential Bell partition polynomial, as defined in chapter 11 of Charalambides [7], is given by the sum

$$B_n(x_1, x_2, \ldots, x_n) = \sum_{\substack{k_1+2k_2+\ldots+nk_n=n \\ k_i \geq 0}} \frac{n!}{k_1(1!)^{k_1}k_2!(2!)^{k_2}\cdots k_n!(n!)^{k_n}} x_1^{k_1}x_2^{k_2}\cdots x_n^{k_n}.$$

[2] see [22, 23, 1, 5, 6]

The generating function for the exponential Bell partition polynomial is

$$\sum_{n=0}^{\infty} B_n(x_1,\ldots,x_n)\frac{t^n}{n!} = \exp\left(\sum_{j=1}^{\infty} x_j \frac{t^j}{j!}\right).$$

One our most important result is to show how to find closed formulas for the number of k-colored partitions, the number of representations of n as a sum of squares and further, using Bell polynomials. In order to to this, the sigma function, has great importance for our purposes. For a complex number j, the arithmetical function σ_j is defined by:

$$\sigma_j(n) = \sum_{d|n} d^j.$$

In the next section we will obtain the next result ([1]) as a particularly case:

$$p_k(n) = \sum_{l=1}^{n} \frac{k^l}{l!}\left(\sum_{(\omega_1,\ldots,\omega_l)\in C_n} \frac{\sigma_1(\omega_1)\sigma_1(\omega_2)\cdots\sigma_1(\omega_l)}{\omega_1\omega_2\cdots\omega_l}\right),$$

for $n > 0$.

2 Main result and applications

We start this section with our main result.

Theorem 2.1. *If $F(z)$ and $G(z)$ are the Z-transforms of the sequences $\{f_k(0), f_k(1), f_k(2), \ldots\}$ and $\{0, kg(1), \ldots\}$, respectively, then*

$$f_k(n) = \frac{1}{n!}B_n\Big(0!kg(1), 1!kg(2), 2!kg(3), \ldots, (n-1)!kg(n)\Big).$$

Proof. Since

$$F(z) = 1 + \frac{f_k(1)}{z} + \frac{f_k(2)}{z^2} + \cdots, \quad G(z) = \frac{kg(1)}{z} + \frac{kg(2)}{z^2} + \cdots, \tag{2.1}$$

then (1.1) gives the differential equation:

$$\frac{d}{dz}F = -\frac{G(z)F(z)}{z}, \tag{2.2}$$

and integrating both sides of (2.2), we have:

$$\mathrm{Ln}(F) = \frac{kg(1)}{z} + \frac{kg(2)}{2z^2} + \frac{kg(3)}{3z^3} + \cdots, \tag{2.3}$$

that is:

$$F(z) := \sum_{n=0}^{\infty} f_k(n) \frac{1}{z^n} = \exp\left(\sum_{j=1}^{\infty} \frac{kg(j)}{j} \frac{1}{z^j}\right). \tag{2.4}$$

On the other hand, we have the generating function of the complete Bell polynomials:

$$\sum_{n=0}^{\infty} \frac{1}{n!} B_n\left(x_1, x_2, \ldots, x_n\right) \frac{1}{z^n} = \exp\left(\sum_{j=1}^{\infty} \frac{x_j}{j!} \frac{1}{z^j}\right) \tag{2.5}$$

whose comparison with (2.4) implies (1.2).

□

The equation (1.2) shows that $f_k(n)$ is a polynomial in the variable k of degree n:

$$f_k(n) = g(n,n)k^n + g(n,n-1)k^{n-1} + \cdots + g(n,2)k^2 + g(n,1)k = \sum_{j=1}^{n} g(n,j)k^j. \tag{2.6}$$

From (2.4):

$$\sum_{n=0}^{\infty} f_k(n)u^n = \sum_{j=0}^{\infty} \frac{k^j}{j!} \left(\frac{g(1)}{1}u + \frac{g(2)}{2}u^2 + \frac{g(3)}{3}u^3 + \cdots\right)^j, \tag{2.7}$$

so, it is not difficult to see that (2.6) and (2.7) imply the interesting expression:

$$g(n,j) = \frac{1}{j!} \sum_{(\omega_1,\ldots,\omega_j)\in\mathcal{C}_n} \frac{g\left(\omega_1\right)\cdots g\left(\omega_j\right)}{\omega_1\cdots\omega_j}, \quad j = 1, 2, \ldots, n, \tag{2.8}$$

for the coefficients of the polynomial (2.6) in terms of the set $\mathcal{C}_n$ of compositions of n and the corresponding values $g\left(\omega_r\right)$; hence from (1.2), (2.6) and (2.8):

$$B_n\Big(0!kg(1), 1!kg(2), 2!kg(3), \ldots, (n-1)!kg(n)\Big) = \sum_{j=1}^{n} \frac{n!}{j!} k^j \sum_{(\omega_1,\ldots,\omega_j)\in\mathcal{C}_n} \frac{g\left(\omega_1\right)\cdots g\left(\omega_j\right)}{\omega_1\cdots\omega_j}. \tag{2.9}$$

For example, this property (2.9) with $k = 1$ and $g(m) = m$ gives an identity for the Lah numbers [15, 10, 4, 24, 19, 20, 25]:

$$B_n\big(1!, 2!, 3!, \ldots, n!\big) = \sum_{j=1}^{n} \frac{n!}{j!} \binom{n-1}{j-1} = \sum_{j=1}^{n} L_n^{[j]}. \tag{2.10}$$

We know the following recurrence relations [26, 27, 21, 28, 29, 3, 37, 32] with the structure (1.1):

$$np_k(n) = k\sum_{j=1}^{n}\sigma_1(j)p_k(n-j), \tag{2.11}$$

$$nr_k(n) = k\sum_{j=1}^{n}2(-1)^{j-1}jD(j)r_k(n-j), \quad D(j) = \sum_{odddlj}\frac{1}{d}, \tag{2.12}$$

$$nt_k(n) = k\sum_{j=1}^{n}T(j)t_k(n-j), \quad T(j) = \sum_{dlj}(-1)^{d-1}d, \tag{2.13}$$

where σ_1 is the sum of divisors function, [44, 2, 31, 40, 43], $r_k(n)$, and $t_k(n)$ are the number of representations of n as a sum of squares and as a sum of triangular numbers, respectively [3, 46, 30, 35]. Then from (1.2), (2.6) and (2.8):

$$p_k(n) = \frac{1}{n!}B_n\Big(0!k\sigma_1(1), 1!k\sigma_1(2), \ldots, (n-1)!k\sigma_1(n)\Big) = \sum_{j=1}^{n}\frac{k^j}{j!}\sum_{(\omega_1,\ldots,\omega_j)\in\mathcal{C}_n}\frac{\sigma_1(\omega_1)\cdots\sigma_1(\omega_j)}{\omega_1\cdots\omega_j}, \tag{2.14}$$

$$\begin{aligned} r_k(n) &= \frac{1}{n!}B_n\left(2kD(1), -4kD(2), 12kD(3), \ldots, 2(-1)^{n-1}n!kD(n)\right), \\ &= \sum_{j=1}^{n}\frac{(-1)^{n-j}2^j}{j!}k^j\sum_{(\omega_1,\ldots,\omega_j)\in\mathcal{C}_n}D(\omega_1)\cdots D(\omega_j), \end{aligned} \tag{2.15}$$

$$t_k(n) = \frac{1}{n!}B_n\Big(0!kT(1), 1!kT(2), \ldots, (n-1)!kT(n)\Big) = \sum_{j=1}^{n}\frac{k^j}{j!}\sum_{(\omega_1,\ldots,\omega_j)\epsilon C_n}\frac{T(\omega_1)\cdots T(\omega_j)}{\omega_1\cdots\omega_j}, \tag{2.16}$$

thus, this polynomial (2.14) recently obtained by Alegri [1] is a particular case of our general results.

For an additional application, Robbins [38, 39] deduced the recurrence relation:

$$np_D(n) = \sum_{j=1}^{n}\sigma_O(j)p_D(n-j), \quad \sigma_O(n) = \sum_{odd\ d|n}d = \sum_{d|n}(-1)^{d-1}\frac{n}{d}, \tag{2.17}$$

where $p_D(n)$ is the number of partitions of n using only distinct parts, and $\sigma_{\mathcal{O}}(n)$ is sum of the odd positive divisors of n. From ((1.2)), with $k=1$, we get:

$$p_D(n) = \frac{1}{n!}B_n\Big(0!\sigma_{\mathcal{O}}(1), 1!\sigma_{\mathcal{O}}(2), 2!\sigma_{\mathcal{O}}(3), \ldots, (n-1)!\sigma_{\mathcal{O}}(n)\Big). \tag{2.18}$$

Remark 2.2. We know the identity $B_n\big(0!, 1!, 2!, \ldots, (n-1)!\big) = n$, then (2.9) with $k = g(m) = 1$ gives a property satisfied by the set of compositions of n :

$$\sum_{j=1}^{n} \frac{1}{j!} \sum_{(\omega_1,\ldots,\omega_j)\in\mathcal{C}_n} \frac{1}{\omega_1\cdots\omega_j} = 1, \quad n \geq 1. \tag{2.19}$$

Remark 2.3. The Bell numbers (see [45, 33, 34]) can be written in the form $B(n) = B_n(1, 1, \ldots, 1)$, thus (2.9) with $k = 1$ and $g(m) = \frac{1}{(m-1)!}$ implies an alternative manner to define these numbers:

$$B(n) = n! \sum_{j=1}^{n} \frac{1}{j!} \sum_{(\omega_1,\ldots,\omega_j)\in\mathcal{C}_n} \frac{1}{\omega_1!\omega_2!\cdots\omega_j!}. \tag{2.20}$$

Using the definition of the complete Bell polynomials we can state the next identity.

Corollary 2.4. For $n > 0$, and $x_i = (i-1)!k\sigma_1(i)$, $1 \leq i \leq n$, we have

$$\frac{B(n)(x_1, x_2, \ldots, x_n)}{n!} = p_k(n)$$

$$= \sum_{\substack{k_1+2k_2+\ldots+nk_n=n \\ k_i\geq 0}} \frac{k^{k_1+k_2+\ldots+k_n}(\sigma_1(1))^{k_1}(\sigma_1(2))^{k_2}\cdots(\sigma_1(n))^{k_n}}{(2!)^{k_2-k_3}(3!)^{k_3-k_4}\cdots((n-1)!)^{k_{n-1}-k_n}(n!)^{k_n}k_1!k_2!\cdots k_n!}.$$

3 Conclusion

Just as finding formulas for colored partitions using the Bell polynomial was possible, a reasonable question is whether there is a way to find a formula for the number of other classes of partitions different from those explored here. An example is plane partitions (sequence A000219 on Oeis[36]), whose generating function for the number of such partitions is given by the following product:

$$y = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^n},$$

where $q \in \mathbb{C}$, $|q| < 1$.

We believe that results involving Bell polynomials and many others can be obtained by interested mathematicians as done here.

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Received in 10 October 2024.
Accept in 12 October 2024.