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ON THE MERCA'S CONNECTION BETWEEN THE PARTITION FUNCTION AND EULER'S TOTIENT

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Abstract

We employ the Z-transform to study a result of Merca involving the partition function $p(n)$ and Euler's totient. Besides, we obtain an identity valid for pentagonal numbers and an arbitrary prime number.

Keywords: Partition function, Dirichlet character, Euler totient function, Z-transform, Möbius function.

1 Introduction

In [1], Merca obtained the next equation:

$$\sum_{k=1}^{n} kp(n-k) = \sum_{k=1}^{n} \phi(k) S_{n,k}, \tag{1.1}$$

which is a connection between the partition and Euler totient functions,[1] where $S_{n,k}$ is the number of $k's$ in all partitions of n. Similarly in [1], the authors found:

$$p(n) = \sum_{k=1}^{n+1} \mu(k) S_{n+1,k}, \tag{1.2}$$

[1]More information about integer partitions and Euler totient function can be found in [3, 4, 5, 6, 2].

with the participation of the Möbius function [2]. The relations (1.1) and (1.2) provide remarkable expressions connecting a function of multiplicative number theory with one of additive number theory.

Besides, Merca [1], established the result:

$$\sum_{k=1}^{n} g(k)p(n-k) = \sum_{k=1}^{n} f(k)S_{n,k}, \tag{1.3}$$

for an arbitrary arithmetic function f and $g(n) = \sum_{d|n} f(d)$. Here we use the Z-transform to show that (1.1) and (1.3) allows to deduce the identities:

$$p(n) = \sum_{k=1}^{n+1} \phi(k)(S_{n+1,k} - 2S_{n,k} + S_{n-1,k}), \tag{1.4}$$

and

$$\sum_{r=j}^{p} a_{p-r}S_{r,j} = \begin{cases} 1, & \text{if } j = 1, p; \\ 0, & \text{if } 2 \leq j \leq p-1, \end{cases} \tag{1.5}$$

for $p = 2, 3, 5, 7, 11, \ldots$, where:

$$a_j = \begin{cases} 0, & \text{if } j \neq \frac{m(3m+1)}{2}; \\ (-1)^m, & \text{if } j = \frac{m(3m+1)}{2}, \end{cases} \tag{1.6}$$

for $m \in \mathbb{Z}$.

One may note that $(a_j)_{j\in\mathbb{Z}}$ satisfy the Euler pentagonal number theorem, as follows.

$$\prod_{n=1}^{\infty}(1 - x^n) = \sum_{n=-\infty}^{\infty} (-1)^k x^{k(3k-1)/2}.$$

The inverse of the previous infinity product gives the generating function for the sequence $(p(n))_{n\in\mathbb{N}\cup\{0\}}$.

The Z-transform is a mathematical tool used in discrete-time signal processing and control theory. It converts discrete-time signals (which are sequences) into a complex frequency domain representation. It is the discrete counterpart of the Laplace Transform, which is used for continuous-time systems, as explained in [9, 10].

[2]In the references [3, 7, 8, 15, 16, 17] one can be found several results involving the Möbius function.

Definition 1.1. For a sequence $(x_n)_{n\in\mathbb{N}\cup 0}$ its Z-transform $X(z)$ is defined as:

$$\mathcal{Z}\{x_n\} = X(z) = \sum_{n=0}^{\infty} x_n z^{-n},$$

where z is a complex variable.

Example 1.2. For $u_n = 1$, $\forall n \in \mathbb{N}\cup 0$, its Z-transform is

$$\mathcal{Z}\{u_n\} = U(z) = \sum_{n=0}^{\infty} z^{-n} = \frac{z}{z-1},$$

which is convergent for $|z| > 1$

Example 1.3. For $x_n = n$, $\forall n \in \mathbb{N}\cup\{0\}$, its Z-transform is

$$\mathcal{Z}\{x_n\} = X(z) = \sum_{n=0}^{\infty} z^{-n} = \frac{z}{(z-1)^2},$$

which is convergent for $|z| > 1$.

In the previous example, it is customary to denote $\mathcal{Z}\{x_n\}$ by $\mathcal{Z}\{n\}$.
We will list some properties of the Z-transform useful to obtain our results.

- [Linearity] $\mathcal{Z}\{x_n \pm y_n\} = \mathcal{Z}\{x_n\} \pm \mathcal{Z}\{y_n\}$
- [Multiplication by a constant] $\mathcal{Z}\{ax_n\} = a\mathcal{Z}\{x_n\}$
- [Convolution] $\mathcal{Z}\{\sum_{k=0}^{n} x_k y_{n-k}\} = \mathcal{Z}\{x_n\}\mathcal{Z}\{y_n\}$

The Z-transform $X(z)$ of a given sequence $(x_n)_{n\in\mathbb{N}\cup\{0\}}$ is unique. The inverse Z-transform is

$$x_n = \mathcal{Z}^{-1}\{X(z)\} = \frac{1}{2\pi i}\oint_C X(z)z^{n-1}dz,$$

where C is a counterclockwise closed path encircling the origin and entirely in the region of convergence of $X(z)$.

2 Main Results

Theorem 2.1. *For all $n \geq 0$, we have*

$$p(n) = \sum_{k=1}^{n+1} \phi(k)(S_{n+1,k} - 2S_{n,k} + S_{n-1,k}).$$

Proof. Denoting by

$$q_n = \sum_{k=1}^{n} kp(n-k) = \sum_{k=1}^{n} \phi(k)S_{n,k},$$

and considering $\sum_{k=1}^{n} kp(n-k)$ as a Cauchy convolution, we have

$$\mathcal{Z}\{q_n\} = \mathcal{Z}\{p(n)\}\mathcal{Z}\{k\},$$

thus

$$\mathcal{Z}\{p(n)\} = \frac{(z-1)^2}{z}\mathcal{Z}\{q_n\},$$

by example 2.

Therefore

$$\mathcal{Z}\{p(n)\} = \mathcal{Z}\{q_{n+1} - 2q_n + q_{n-1}\},$$

which implies in the equation (1.4). □

Theorem 2.2.

$$\sum_{r=j}^{p} a_{p-r}S_{r,j} = \begin{cases} 1, & \text{if } j = 1, p; \\ 0, & \text{if } 2 \leq j \leq p-1, \end{cases}$$

for p a prime number, where a_j is given by (1.6).

Proof. Applying Z-transform in both sides of (1.3), we have:

$$\mathcal{Z}\{\sum_{k=1}^{n} g(k)p(n-k)\} = \mathcal{Z}\{g(n)\}\mathcal{Z}\{p(n)\} = \mathcal{Z}\{\sum_{k=1}^{n} f(k)S_{n,k}\}.$$

By the Pentagonal number theorem of Euler, we have:

$$\left(\sum_{n=0}^{\infty} p(n)q^n\right)\left(\prod_{n=1}^{\infty}(1-q^n)\right) = 1,$$

and using the Z-transform, we can obtain

$$\mathcal{Z}\{p(n)\} = \left(\mathcal{Z}\{a_n\}\right)^{-1}.$$

Thus,

$$\mathcal{Z}\{g(n)\} = \mathcal{Z}\{a_n\}\mathcal{Z}\{\sum_{k=1}^{n} f(k)S_{n,k}\},$$

and using the convolution property again, we have:

$$g(n) = \sum_{d|n} f(d) = \sum_{k=1}^{n} f(k)S_{n,k}, n \geq 1, \tag{2.1}$$

for any arithmetic function f. If n is a prime number, then $g(p) = f(1) + f(p)$ and from (2.1), we can conclude the proof.

□

Prof. Merca ([11]) comments that the previous result could be a particular case of general results obtained in [12].

If $f = J_m$ is the Jordan totient function as defined in [3], thus $g(n) = n^m$ and from equation (1.3), we have:

$$\sum_{k=1}^{n} k^m p(n-k) = \sum_{k=1}^{n} J_m(k)S_{n,k}.$$

For $m = 2$, using the Z-transform as did in the last theorem, we can get the result as follows.

Theorem 2.3. *For $n \geq 0$,*

$$p(n) = J_2(n+1)+J_2(n)(S_{n+1,n}-4)+\sum_{k=1}^{n-1} J_2(k)\left(S_{n+1,k} - 4S_{n,k} - S_{n-1,k} - 8\sum_{j=k}^{n-1}(-1)^(n-j)S_{j,k}\right)$$

An interesting application of (1.3) is for f is the nontrivial Dirichlet character (mod 4), as defined in [13, 14]:

$$f(n) = \chi_4(n) = \begin{cases} (-1)^{\frac{n-1}{2}}, & \text{if } n \text{ is odd;} \\ 0, & \text{if } n \text{ is even} \end{cases} = \begin{cases} 1, & \text{if } n \equiv 1(mod4); \\ -1, & \text{if } n \equiv -1(mod4); \\ 0, & \text{if } n \equiv 0(mod2), \end{cases}$$

and $g(n) = \sum_{d|n} \chi_4(d) = \frac{1}{4} r_2(n)$, by the Jacobi's identity as given in [13, 18]. Using the aforementioned identity (1.3), we have the result as follows.

Corollary 2.4. For $n \geq 1$, we have:

$$\sum_{k=1}^{n} r_2(k) p(n-k) = 4 \sum_{k=1}^{n} \chi_4(k) S_{n,k}.$$

Applying Z-transform in the previous equation, we have:

$$\mathcal{Z}\{\sum_{k=1}^{n} r_2(k) p(n-k)\} = \mathcal{Z}\{r_2(n)\}\mathcal{Z}\{p(n)\} = \mathcal{Z}\{4 \sum_{k=1}^{n} \chi_4(k) S_{n,k}\}.$$

As we known,

$$\mathcal{Z}\{p(n)\} = \left(\mathcal{Z}\{a_n\}\right)^{-1},$$

then

$$\mathcal{Z}\{r_2(n)\} = \mathcal{Z}\{a_n\}\mathcal{Z}\{4 \sum_{k=1}^{n} \chi_4(k) S_{n,k}\}.$$

For

$$b_n = 4 \sum_{k=1}^{n} \chi_4(k) S_{n,k}, \tag{2.2}$$

we have:

$$\mathcal{Z}\{r_2(n)\} = \mathcal{Z}\{\sum_{l=1}^{n} a_{n-l} b_l\}.$$

We can conclude this section with the next result.

Theorem 2.5. *For $n \geq 1$,*

$$r_2(n) = \sum_{l=1}^{n} a_{n-l} b_l,$$

where b_n is given as (2.2), and a_n as (1.6).

3 Conclusion

Our analysis shows that the Z-transform is useful to study relations involving arithmetic functions, in particular, its application to results obtained by Merca [1] allows to deduce the explicit expression (1.4) for the partition function $p(n)$ in terms of Euler's totient, and the identity (1.5) for pentagonal numbers. The result of Theorem 2.3 is nontrivial and it connects $p(n)$ with the Jordan's totient $J_2(n)$.

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