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MOCK THETA FUNCTIONS AS GENERATING FUNCTIONS FOR CERTAIN PLANE PARTITIONS

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Abstract

This paper provides a combinatorial interpretation of mock theta functions as generating functions for certain classes of plane partitions through a uniform procedure.

Keywords: Congruence, cubic partition, 3-core partition.

1 Introduction

In his last letter to Hardy in 1920 (see [13, p. 127–131]), Ramanujan introduced the notion of a mock theta function. He listed 17 such functions having orders 3, 5, and 7. Since then, the mock theta functions have received much attention. In addition to the classical mock theta functions, many new ones have been discovered recently, see [9, 11, 12] for example.

Combinatorial aspects of mock theta functions have been investigated by many authors, including [1, 2, 4, 7, 15, 16]. For instance, in [7] the mock theta functions are combinatorially interpreted in terms of two-line arrays. Combinatorial interpretation in terms of partitions can be seen in [1, 2, 8, 15, 16]. For example, Choi and Kim [8] provide partition theoretic properties of third order mock theta functions $\phi(q)$, $\psi(q)$, $\nu(q)$ and sixth order mock theta functions $\Psi(q)$, $\Psi_-(q)$, $\rho(q)$, and $\lambda(q)$ in terms of n-color partitions and n-color overpartitions. Choi and Kim close their paper noting that it would be interesting to see a description of the mock theta functions as generating functions for certain classes of plane partitions or plane overpartitions.

We recall that a plane partition of the positive integer n is an array of non-negative integers

$$\begin{matrix} n_{11} & n_{12} & n_{13} & \cdots \\ n_{21} & n_{22} & n_{23} & \cdots \\ \vdots & \vdots & \vdots & \cdots \end{matrix}$$

for which $\sum n_{ij} = n$ and the rows and columns are arranged in non-increasing order: $n_{ij} \geq n_{(i+1)j}$ and $n_{ij} \geq n_{i(j+1)}$, for all $i, j \geq 1$.

The goal of this paper is to provide combinatorial interpretations in terms of plane partitions for the mock theta functions, which answers the question raised by Choi and Kim [8] about describing the mock theta functions as generating functions for certain classes of plane partitions.

This paper is organized as follows. In Section 2, we describe a class of plane partitions and how it can be obtained from a two-line matrix. Section 3 is devoted to proving the main results of this paper. We close the paper summarizing the combinatorial interpretations for the mock theta functions in Section 4.

2 Preliminaries

In this section, we describe two important constructions that will be used throughout the paper. The first one associates a given three-line matrix to a lattice path, which is linked to a plane partition in a unique way. The second one describes how we transform certain two-line matrices into three-line matrices. The last construction is essential for building the plane partitions from mock theta functions thanks to what was proven in [7].

2.1 A special class of plane partitions

Given a three-line matrix consisting of positive integers, we describe below how we can associate it to a lattice path in the 3-dimensional space, to build a volume, which is going to correspond to a plane partition. This construction can be done in many different ways, each one providing us with a family of plane partitions. However, as we are seeking a uniform class of plane partitions generated by mock theta functions, we choose a specific way to do this construction.

Consider, for example, the three-line matrix

$$\begin{pmatrix} 12 & 10 & 7 & 2 \\ 2 & 3 & 1 & 4 \\ 5 & 1 & 2 & 1 \end{pmatrix}. \tag{2.1}$$

To obtain a solid representing a plane partition from (2.1), we draw the lattice path starting at $(0, 1, 1)$ moving 12 units in the x-direction, 2 units in the y-direction, 5 units in the z-direction, 10 units in the x-direction, 3 units in the y-direction, 1 unit in the z-direction, and so on. At the end, we insert an extra move of 1 unit in the x-direction.

In this way, we generate a unique 3-dimensional path. The corresponding path is shown in Figure 1, where a $*$ together with a number near a corner represents a step in the positive z-direction.

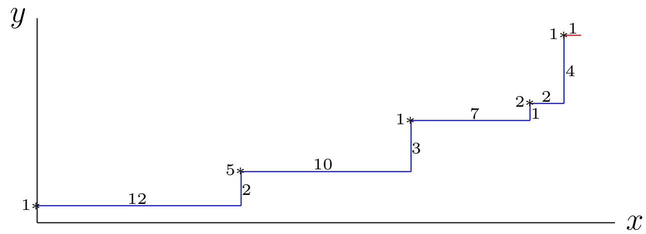

Figure 1: The lattice path associated to matrix (2.1)

From the path shown in Figure 1, we need to get a plane partition in the conventional way, i.e., as a solid in the first octant. In order to do this, we apply the transformation $x \mapsto -x$, which corresponds to a reflection concerning the yz-plane followed by a translation of 32 units (the sum of the entries in the first line of the matrix (2.1) plus one) in the x-direction. The resulting lattice path is shown in Figure 2.

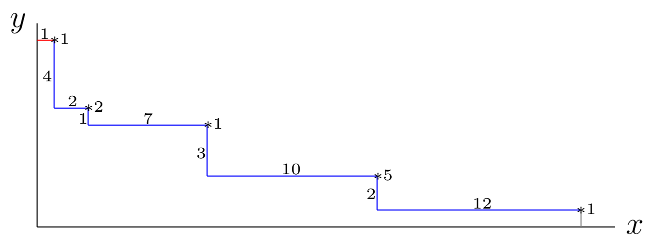

Figure 2: The lattice path that gives the plane partition associated to matrix (2.1)

Now, we construct the plane partition by stacking $1 \times 1 \times 1$ cubes into the solid limited by the lattice path shown in Figure 2, respecting the high of the levels in the z-direction. The plane partition corresponding to this solid is shown in Figure 3. Note that the number of levels parallel to each of the planes xy, xz, and yz is the same.

The type of plane partitions that are relevant to the rest of the paper is defined below.

Definition 2.1. A plane partition is said to be of type $\mathcal{R}$ if in its geometrical representation, the entries having the same value form a unique rectangle parallel to the

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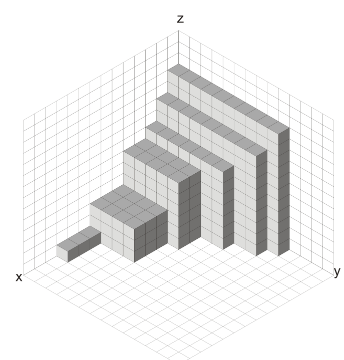
10	9	9	7	7	7	7	7	7	7	6	6	6	6	6	6	6	6	6	6	1	1	1	1	1	1	1	1	1	1	1	1

Figure 3: The plane partition associated to the matrix (2.1)

xy-plane and the number of distinct levels that are parallel to each of the planes xy, xz, and yz is the same.

Considering the construction above, it is easy to write the matrix with positive entries corresponding to a plane partition of type $\mathcal{R}$. To get the entries on the third line, we have to stand on the plane $z = 1$ (since we have started drawing the lattice path at $(0, 1, 1)$), climb to the top, and set the height of those steps as the entries. To get the entries on the second line, we have to stand on the plane $y = 1$, go up to the top, and set the height of those steps as the entries. As we have made a reflection in the construction of the solids, to obtain the entries on the first line, we have to stand on the plane $x = 1$, climb to the top, count the height of those steps, and, then, set the entries as being these integers in reverse order. For instance, in the figure to the right, we have a type $\mathcal{R}$ plane partition and its corresponding three-line matrix.



$$\begin{pmatrix} 3 & 3 & 2 & 1 & 1 \\ 3 & 1 & 2 & 2 & 1 \\ 2 & 3 & 1 & 2 & 2 \end{pmatrix}$$

The general appearance of a type $\mathcal{R}$ plane partition is shown in Figure 4.

We let λ_i be the value of the parts at each level parallel to the xy-plane, while x_j and y_j are the steps in the x-direction and y-direction, respectively. For example, the plane partition shown in Figure 3 has $s = 4$, $\lambda_1 = 1$, $\lambda_2 = 6$, $\lambda_3 = 7$, $\lambda_4 = 9$, $\lambda_5 = 10$, $x_1 = 12$, $x_2 = 10$, $x_3 = 7$, $x_4 = 2$, $x_5 = 1$, $y_1 = 1$, $y_2 = 2$, $y_3 = 3$, $y_4 = 1$, and $y_5 = 4$.

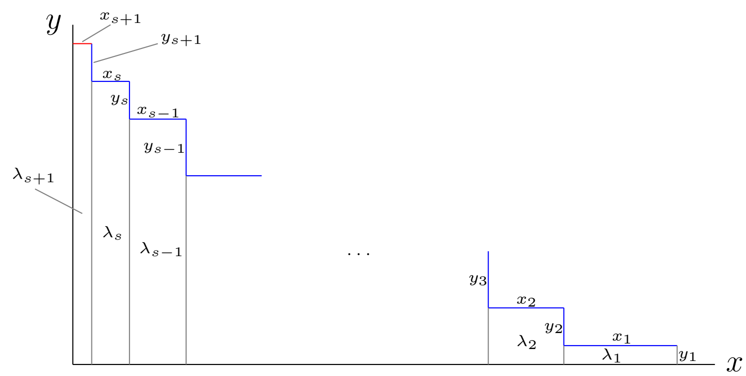


Figure 4: A lattice path of a type $\mathcal{R}$ plane partition

2.2 From two-line and three-line matrices

In [14] three characterizations of unrestricted partitions in terms of two-line matrices are presented. We recall two of them below to illustrate how one can obtain a three-line matrix from a two-line matrix in such a way that a plane partition can be built.

Theorem 2.2 (Theorem 8, [14])**.** *The number of unrestricted partitions of n is equal to the number of two-line matrices of the form*

$$\begin{pmatrix} c_1 & c_2 & c_3 & \dots & c_s \\ d_1 & d_2 & d_3 & \dots & d_s \end{pmatrix}, \tag{2.2}$$

where

$$\begin{aligned} & c_s = 0, \\ & c_t = c_{t+1} + d_{t+1}, \ \forall t < s, \\ & n = \sum c_t + \sum d_t. \end{aligned} \tag{2.3}$$

Theorem 2.3 (Theorem 10, [14])**.** *The number of unrestricted partitions of n is equal to the number of two-line matrices of the form*

$$\begin{pmatrix} c_1 & c_2 & c_3 & \dots & c_s \\ d_1 & d_2 & d_3 & \dots & d_s \end{pmatrix}, \tag{2.4}$$

where

$$\begin{aligned} & c_s \neq 0, \\ & c_t \geq 2 + c_{t+1} + d_{t+1}, \ \forall t < s, \\ & n = \sum c_t + \sum d_t. \end{aligned} \tag{2.5}$$

Bijective proofs of these theorems can be found in [5].

We now discuss how we will associate a three-line matrix to a given matrix of one of the types appearing in the above theorems. Consider a two-line matrix (2.2) satisfying (2.3), namely,

$$\begin{pmatrix} c_1 & c_2 & c_3 & \cdots & c_s \\ d_1 & d_2 & d_3 & \cdots & d_s \end{pmatrix}, \tag{2.6}$$

where

$$\begin{array}{l} c_t = a + c_{t+1} + d_{t+1}, \ \forall t < s, \\ n = \sum c_t + \sum d_t, \end{array}$$

with a positive integer and c_s is a given constant. Initially, we add a third line with zeros in (2.2). In order to obtain a type $\mathcal{R}$ plane partition, we add 1 to each entry of the resulting matrix, obtaining

$$\begin{pmatrix} a(s-1) + c_s + d_2 + \cdots d_s + 1 & \cdots & a + c_s + d_s + 1 & c_s + 1 \\ d_1 + 1 & \cdots & d_{s-1} + 1 & d_s + 1 \\ 1 & \cdots & 1 & 1 \end{pmatrix}. \tag{2.7}$$

Matrix (2.7) is what we need to build the lattice path and, then, the plane partition as we discussed previously.

As another example, consider a matrix (2.4) satisfying (2.5), namely

$$\begin{array}{l} c_t \geq a + c_{t+1} + d_{t+1}, \ \forall t < s, \\ n = \sum c_t + \sum d_t, \end{array} \tag{2.8}$$

with a a positive integer and c_s is a given constant. From the restrictions (2.8), we see that there exists, for each t, a non-negative integer i_t such that $c_t = a + i_t + c_{t+1} + d_{t+1}$, for $1 \leq t \leq s-1$, and $i_s = c_s$. Hence, we can rewrite matrix (2.4) as

$$\begin{pmatrix} a(s-1) + i_1 + \cdots + i_s + d_2 + \cdots + d_s & \cdots & a + i_{s-1} + i_s + d_s & i_s \\ d_1 & \cdots & d_{s-1} & d_s \end{pmatrix}. \tag{2.9}$$

We associate a three-line matrix to (2.9) by subtracting i_t from the t-th entry in the first row and setting i_t as the t-th entry of a new third row:

$$\begin{pmatrix} a(s-1) + i_2 + \cdots + i_s + d_2 + \cdots + d_s & \cdots & a + i_s + d_s & 0 \\ d_1 & \cdots & d_{s-1} & d_s \\ i_1 & \cdots & i_{s-1} & i_s \end{pmatrix}. \tag{2.10}$$

Note that matrix (2.10) may have zero entries. Then, in order to obtain a type $\mathcal{R}$ plane partition, we add 1 to each entry of this matrix, obtaining:

$$\begin{pmatrix} a(s-1) + 1 + i_2 + \cdots + i_s + d_2 + \cdots + d_s & \cdots & a + 1 + i_s + d_s & 1 \\ 1 + d_1 & \cdots & 1 + d_{s-1} & 1 + d_s \\ 1 + i_1 & \cdots & 1 + i_{s-1} & 1 + i_s \end{pmatrix}. \tag{2.11}$$

This procedure can be easily reversed to get (2.4) again.

This method permits us to associate a finite set of restricted partitions of n to a finite set of three-line matrices of the form

$$\begin{pmatrix} c_1 & c_2 & c_3 & \cdots & c_s \\ d_1 & d_2 & d_3 & \cdots & d_s \\ e_1 & e_2 & e_3 & \cdots & e_s \end{pmatrix}, \tag{2.12}$$

where $e_j = \lambda_{j+1} - \lambda_j$, $x_j = c_j$ and $d_j = y_{j+1}$, for $j = 1, \dots, s$, considering the preceding notation of λ_j, x_j and y_j.

3 The plane partitions generated by mock theta functions

Our goal in this section is to interpret the mock theta functions as generating functions for certain type $\mathcal{R}$ plane partitions. The idea is to employ the combinatorial interpretation as two-line matrices for the mock theta functions from [7] to create three-line matrices and, then, build the plane partitions, proceeding according to the method presented in the last sections.

We use the notation $(a;q)_0 = 1$ and $(a;q)_n = \prod_{k=0}^{n-1}(1 - aq^k)$ for any positive integer n.

Consider the third-order mock theta function:

$$f(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n^2}.$$

In [7] this mock theta function was shown to be the generating function for two-line matrices of the form

$$\begin{pmatrix} c_1 & c_2 & c_3 & \cdots & c_s \\ d_1 & d_2 & d_3 & \cdots & d_s \end{pmatrix} \tag{3.1}$$

with non-negative integer entries satisfying

$$\begin{aligned} & c_s \geq 1, \\ & c_t \geq 2 + c_{t+1} + d_{t+1}, \\ & n = \sum c_t + \sum d_t, \end{aligned} \tag{3.2}$$

and weight $(-1)^{d_1 - c_1 + 1}$. From the restrictions in (3.2), we see that there are, for each t, a non-negative integer i_t such that $c_t = 2 + i_t + c_{t+1} + d_{t+1}$, for $1 \leq t \leq s-1$, and

$c_s = 1 + i_s$. Hence, we can rewrite matrix (3.1) as:

$$\begin{pmatrix} 2s-1+i_1+\cdots+i_s+d_2+\cdots+d_s & \cdots & 3+i_{s-1}+i_s+d_s & 1+i_s \\ d_1 & \cdots & d_{s-1} & d_s \end{pmatrix}. \tag{3.3}$$

We associate a three-line matrix to (3.3) by subtracting i_t from the t-th entry in the first row and setting i_t as the t-th entry of a new third row:

$$\begin{pmatrix} 2s-1+i_2+\cdots+i_s+d_2+\cdots+d_s & \cdots & 3+i_s+d_s & 1 \\ d_1 & \cdots & d_{s-1} & d_s \\ i_1 & \cdots & i_{s-1} & i_s \end{pmatrix}. \tag{3.4}$$

Note that matrix (3.4) may have zero entries. Then, in order to obtain a type $\mathcal{R}$ plane partition, we add 1 to each entry of this matrix, obtaining:

$$\begin{pmatrix} 2s+i_2+\cdots+i_s+d_2+\cdots+d_s & \cdots & 4+i_s+d_s & 2 \\ 1+d_1 & \cdots & 1+d_{s-1} & 1+d_s \\ 1+i_1 & \cdots & 1+i_{s-1} & 1+i_s \end{pmatrix}. \tag{3.5}$$

This procedure can be easily reversed to get (3.1) again.

Now we can use the procedure described in the subsection 2.1 to associate a unique type $\mathcal{R}$ plane partition to each matrix (3.5). The next theorem presents the result for the mock theta function $f(q)$. In this theorem, we use the parameters from Figure 4 to describe the plane partitions.

Theorem 3.1. *The mock theta function $f(q)$ is the generating function for type $\mathcal{R}$ plane partitions having weight $(-1)^{y_2-x_1-\lambda_2+\lambda_1}$ and satisfying:*

i. $\lambda_1 = 1,\ \lambda_j - \lambda_{j-1} \geq 1,$

ii. $y_1 = 1,\ y_j \geq 1,$

iii. $x_{s+1} = 1, x_s = 2,\ x_j - x_{j+1} = y_{j+2} + \lambda_{j+2} - \lambda_{j+1}.$

Proof. Firstly we note that the plane partitions obtained from (3.1) have: $y_1 = 1,\ y_j = 1 + d_{j-1}, j = 2, \ldots, s+1,\ \lambda_1 = 1,\ \lambda_{j+1} - \lambda_j = 1 + i_j, j = 1, \ldots, s,$ and

$$\begin{aligned} x_1 &= 2s + i_2 + i_3 + \cdots + i_s + d_2 + d_3 + \cdots + d_s, \\ x_2 &= 2(s-1) + i_3 + \cdots + i_s + d_3 + \cdots + d_s, \\ &\vdots \\ x_{s-1} &= 4 + i_s + d_s, \\ x_s &= 2, \\ x_{s+1} &= 1. \end{aligned}$$

Then, $\lambda_{i+1} - \lambda_i \geq 1$ and $y_i \geq 1$. We also have $x_j - x_{j+1} = 2 + i_{j+1} + d_{j+1} = 1 + i_{j+1} + 1 + d_j = \lambda_{j+2} - \lambda_{j+1} + y_{j+2}$. Finally, the weight of the plane partitions are:

$$\begin{aligned}(-1)^{d_1 - c_1 + 1} &= (-1)^{(y_2-1)-(2s+i_2+\cdots+i_s+d_2+\cdots+d_s+(1+i_1)-2)+1},\\ &= (-1)^{y_2-1-(x_1+\lambda_2-\lambda_1-2)+1},\\ &= (-1)^{y_2-x_1-\lambda_2+\lambda_1}.\end{aligned}$$

Conversely, it is easy to see that, given a type $\mathcal{R}$ plane partition satisfying conditions i, ii, and iii, we can associate to it a unique matrix of the form (3.1). □

Now, consider the third-order mock theta function:

$$\phi(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q^2;q^2)_n}.$$

In [7] this mock theta function was shown to be the generating function for two-line matrices of the form

$$\begin{pmatrix} c_1 & c_2 & c_3 & \cdots & c_s \\ d_1 & d_2 & d_3 & \cdots & d_s \end{pmatrix} \tag{3.6}$$

with non-negative integer entries satisfying

$$\begin{aligned} &c_s = 1,\\ &c_t = 2 + c_{t+1} + d_{t+1},\\ &2 \mid d,\\ &\sum c_t + \sum d_t = n, \end{aligned} \tag{3.7}$$

and weight $(-1)^{\frac{1}{2}\sum_{t=1}^{s} d_t}$.

From the restrictions in (3.7), we can rewrite matrix (3.6) as:

$$\begin{pmatrix} (2s-1) + 2d_2 + \cdots + 2d_s & \cdots & 3 + 2d_s & 1 \\ 2d_1 & \cdots & 2d_{s-1} & 2d_s, \end{pmatrix}. \tag{3.8}$$

We associate a three-line matrix to (3.8) by putting 0 as the t-th entry of a new third row:

$$\begin{pmatrix} (2s-1) + 2d_2 + \cdots + 2d_s & \cdots & 3 + 2d_s & 1 \\ 2d_1 & \cdots & 2d_{s-1} & 2d_s \\ 0 & \cdots & 0 & 0 \end{pmatrix}. \tag{3.9}$$

Note that matrix (3.9) may have zero entries. Then, in order to obtain a type $\mathcal{R}$ plane partition, we add 1 to each entry of this matrix, obtaining:

$$\begin{pmatrix} 2s + 2d_2 + \cdots + 2d_s & \cdots & 4 + 2d_s & 2 \\ 1 + 2d_1 & \cdots & 1 + 2d_{s-1} & 1 + 2d_s \\ 1 & \cdots & 1 & 1 \end{pmatrix}. \tag{3.10}$$

This procedure can be easily reverted to get (3.6) again.

Now we can use the procedure described in the last subsection to associate a unique type $\mathcal{R}$ plane partition to each matrix (3.10). The next theorem presents the result for the mock theta function $\phi(q)$. In this theorem, we use the parameters from Figure 4 to describe the plane partitions.

Theorem 3.2. *The mock theta function $\phi(q)$ is the generating function for type $\mathcal{R}$ plane partitions having weight $(-1)^{\frac{1}{2}(-s+(\sum_{t=2}^{s+1} y_t))}$ and satisfying:*

i. $\lambda_1 = 1,\ \lambda_{j+1} - \lambda_j = 1,$

ii. $y_1 = 1, y_j \equiv 1(mod\ 2),$

iii. $x_{s+1} = 1, x_s = 2, x_j - x_{j+1} = 1 + y_{j+1}.$

Proof. The plane partitions obtained from (3.10) have the parameters $y_1 = 1$, $y_j = 1 + 2d_{j-1}$, then $y_j \equiv 1(\text{mod } 2)$, for $j = 2, \ldots, s+1$, $\lambda_1 = 1$, $\lambda_{j+1} - \lambda_j = 1, j = 1, \ldots, s$, and

$$\begin{array}{lcl} x_1 & = & 2s + 2d_2 + 2d_3 + \cdots + 2d_s, \\ x_2 & = & 2(s-1) + 2d_3 + \cdots + 2d_s, \\ & \vdots & \\ x_{s-1} & = & 4 + 2d_s, \\ x_s & = & 2, \\ x_{s+1} & = & 1. \end{array}$$

Then, we also have $x_j - x_{j+1} = 2 + d_{j+1} = 1 + 1 + i_{j+1} = 1 + y_{j+2}$. Finally, the weight of the plane partitions are:

$$(-1)^{\frac{1}{2}\sum_{t=1}^{s} d_t} \quad = (-1)^{\frac{1}{2}(-s+(\sum_{t=2}^{s+1} y_t))},$$

since $d_j = y_{j+1} - 1$, for $j = 1, \ldots, s$.

Conversely, it is easy to see that, given a type $\mathcal{R}$ plane partition satisfying conditions *i*, *ii*, and *iii*, we can associate to it a unique matrix of the form (3.10). □

Using the analogous idea, consider the third-order mock theta function.

$$\psi(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q^2)_n}.$$

Thus, $\psi(q)$ is the generating function for two-line matrices of the form

$$\begin{pmatrix} c_1 & c_2 & c_3 & \cdots & c_s \\ d_1 & d_2 & d_3 & \cdots & d_s \end{pmatrix} \tag{3.11}$$

with non-negative integer entries satisfying

$$\begin{array}{l} c_s = 1, \\ c_t = 2 + c_{t+1} + 2d_{t+1}, \\ \sum c_t + \sum d_t = n. \end{array} \tag{3.12}$$

The procedure described in the last subsection associates a unique type $\mathcal{R}$ plane partitions given by the following result.

Theorem 3.3. *The mock theta function $\psi(q)$ is the generating function for type $\mathcal{R}$ plane partitions and satisfying:*

i. $\lambda_1 = 1, \lambda_j - \lambda_{j-1} = 1,$

ii. $y_1 = 1, y_j \geq 1,$

iii. $x_{s+1} = 1, x_s = 2, x_j - x_{j+1} = 2y_{j+1}.$

Proof. From the restrictions in (3.12), we can rewrite matrix (3.11) as:

$$\begin{pmatrix} (2s-1) + 2d_2 + \cdots + 2d_s & \cdots & 3 + 2d_s & 1 \\ d_1 & \cdots & d_{s-1} & d_s \end{pmatrix}. \tag{3.13}$$

We associate a three-line matrix to (3.13) by putting 0 as the t-th entry of a new third row:

$$\begin{pmatrix} (2s-1) + 2d_2 + \cdots + 2d_s & \cdots & 3 + 2d_s & 1 \\ d_1 & \cdots & d_{s-1} & d_s \\ 0 & \cdots & 0 & 0 \end{pmatrix}. \tag{3.14}$$

Note that matrix (3.14) may have zero entries. Then, in order to obtain a type $\mathcal{R}$ plane partition, we add 1 to each entry of this matrix, obtaining:

$$\begin{pmatrix} 2s + 2d_2 + \cdots + 2d_s & \cdots & 4 + 2d_s & 2 \\ 1 + d_1 & \cdots & 1 + d_{s-1} & 1 + d_s \\ 1 & \cdots & 1 & 1 \end{pmatrix}. \tag{3.15}$$

Therefore, proceeding as Theorem 3.2 the result is provided. □

The following third-order mock theta function considered is

$$\chi(q) = \sum_{n=0}^{\infty} \frac{(-q;q)_n q^{n^2}}{(-q^3;q^3)_n}.$$

This mock theta function is the generating function for two-line matrices of the form

$$\begin{pmatrix} c_1 & c_2 & c_3 & \cdots & c_s \\ d_1 & d_2 & d_3 & \cdots & d_s \end{pmatrix} \tag{3.16}$$

with non-negative integer entries satisfying

$$\begin{array}{l} c_s \in \{1,2\} \\ c_t = i_t + c_{t+1} + d_{t+1}, i_t \in \{2,3\} \\ 3 \mid d_t \\ \sum c_t + \sum d_t = n \end{array} \tag{3.17}$$

and weight $(-1)^{\frac{1}{3}\sum_{t=1}^{s} d_t}$. Then, the next theorem describes the set of restricted plane partitions that the mock theta function $\chi(q)$ is a generating function.

Theorem 3.4. *The mock theta function $\chi(q)$ is the generating function for type $\mathcal{R}$ plane partitions having weight $(-1)^{\frac{1}{3}(-s+(\sum_{t=2}^{s+1} y_t))}$ and satisfying:*

i. $\lambda_1 = 1$, $3 \leq \lambda_j - \lambda_{j-1} \leq 4$,

ii. $y_1 = 1, y_j \equiv 1(mod\ 3)$,

iii. $x_{s+1} = 1, x_s = 1, x_j - x_{j+1} = \lambda_{j+1} - \lambda_j + y_{j+1} - 2$.

Proof. Recall the interpretation for $\chi(q)$ as the generating functions for the set of two-line matrices in the form (3.17) with restrictions (3.16). From the restrictions in (3.17), we can rewrite matrix (3.16) as:

$$\begin{pmatrix} i_s + i_{s-1} + i_{s-2} + \cdots + i_1 + 3d_2 + \cdots + 3d_s & \cdots & i_s + i_{s-1} + 3d_s & 0 \\ 3d_1 & \cdots & 3d_{s-1} & 3d_s \end{pmatrix}. \tag{3.18}$$

We associate a three-line matrix to (3.18) by subtracting i_t from the t-th entry in the first row and putting i_t as the t-th entry of a new third row:

$$\begin{pmatrix} i_s + i_{s-1} + i_{s-2} + \cdots + i_2 + 3d_2 + \cdots + 3d_s & \cdots & i_s + 3d_s & 0 \\ 3d_1 & \cdots & 3d_{s-1} & 3d_s \\ i_1 & \cdots & i_{s-1} & i_s \end{pmatrix}. \tag{3.19}$$

Note that matrix (3.19) may have zero entries. Then, in order to obtain a type $\mathcal{R}$ plane partition, we add 1 to each entry of this matrix, obtaining:

$$\begin{pmatrix} 1 + i_s + i_{s-1} + i_{s-2} + \cdots + i_2 + 3d_2 + \cdots + 3d_s & \cdots & 1 + i_s + 3d_s & 1 \\ 1 + 3d_1 & \cdots & 1 + 3d_{s-1} & 1 + 3d_s \\ 1 + i_1 & \cdots & 1 + i_{s-1} & 1 + i_s \end{pmatrix}. \tag{3.20}$$

This procedure can be easily reverted to get (3.16) again. Now we can use the procedure described in the last subsection to associate a unique type $\mathcal{R}$ plane partition to each matrix (3.20). The plane partitions obtained from (3.20) have the parameters $y_1 = 1$, $y_j = 1+3d_{j-1}$, then $y_j \equiv 1(\bmod 3)$, for $j = 2, \ldots, s+1$, $\lambda_1 = 1$, $\lambda_{j+1}-\lambda_j = 1+i_j$, since $i_t \in \{2, 3\}$ then $3 \leq \lambda_j - \lambda_{j-1} \leq 4$. for $j = 1, \ldots, s$. Also,

$$\begin{aligned} x_1 &= 1 + i_s + i_{s-1} + i_{s-2} + \cdots + i_2 + 3d_2 + \cdots + 3d_s, \\ &\vdots \\ x_s &= 1 + i_s + 3d_s, \\ x_{s+1} &= 1. \end{aligned}$$

Then, we have $x_j - x_{j+1} = i_j + 3d_j = \lambda_{j+1} - \lambda_j + y_{j+1} - 2$. Finally, the weight of the plane partitions are:

$$(-1)^{\frac{1}{3}\sum_{t=1}^{s} d_t} = (-1)^{\frac{1}{3}(-s+(\sum_{t=2}^{s+1} y_t))},$$

since $d_j = y_{j+1} - 1$, for $j = 1, \ldots, s$.

Conversely, it is easy to see that, given a type $\mathcal{R}$ plane partition satisfying conditions *i*, *ii*, and *iii*, we can associate to it a unique matrix of the form (3.20).

□

The analogous process can be applied to the Third-order mock theta function

$$q\omega(q) = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)+1}}{(q;q^2)_{n+1}^2},$$

generating function for two-line matrices of the form

$$\begin{pmatrix} c_1 & c_2 & c_3 & \cdots & c_s \\ d_1 & d_2 & d_3 & \cdots & d_s \end{pmatrix} \tag{3.21}$$

with non-negative integer entries satisfying

$$\begin{aligned} &c_s = 1 \\ &c_t = c_{t+1} + 2d_{t+1}, \\ &\sum c_t + \sum d_t = n. \end{aligned} \tag{3.22}$$

And also for the third-order mock theta function

$$\nu(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(-q;q^2)_{n+1}},$$

generating function for two-line matrices of the form

$$\begin{pmatrix} c_1 & c_2 & c_3 & \cdots & c_s \\ d_1 & d_2 & d_3 & \cdots & d_s \end{pmatrix} \tag{3.23}$$

with non-negative integer entries satisfying

$$\begin{aligned} & c_s = 0 \\ & c_t = 2 + c_{t+1} + 2d_{t+1} \\ & \sum c_t + \sum d_t = n \end{aligned} \tag{3.24}$$

and weight $(-1)^n$. The following results are the interpretations in terms of plane partitions for them, respectively.

Theorem 3.5. *The mock theta function $q\omega(q)$ is the generating function for type $\mathcal{R}$ plane partitions and satisfying:*

i. $\lambda_1 = 1,\ \lambda_j - \lambda_{j-1} = 1,$

ii. $y_1 = 1, y_j \geq 1,$

iii. $x_{s+1} = 1, x_s = 2, x_j - x_{j+1} = 2(y_{j+1} - 1), x_j \equiv 0 (mod\ 2).$

Theorem 3.6. *The mock theta function $\nu(q)$ is the generating function for type $\mathcal{R}$ plane partitions having weight $(-1)^{\sum_{t=1}^{s} x_t + y_{t+1}}$ and satisfying:*

i. $\lambda_1 = 1,\ \lambda_j - \lambda_{j-1} = 1,$

ii. $y_1 = 1, y_j \geq 1,$

iii. $x_{s+1} = 1, x_s = 1, x_j - x_{j+1} = 2y_{j+2} - 1.$

The next third-order mock theta function treated is

$$\rho(q) = \sum_{n=0}^{\infty} \frac{(-q;q^2)_{n+1} q^{2n(n+1)}}{(-q^3;q^6)_{n+1}}.$$

In [7] this mock theta function was shown to be the generating function for two-line matrices of the form

$$\begin{pmatrix} c_1 & c_2 & c_3 & \cdots & c_s \\ d_1 & d_2 & d_3 & \cdots & d_s \end{pmatrix} \tag{3.25}$$

with non-negative integer entries satisfying

$$\begin{array}{l} c_s \in \{0,1\} \\ c_t = 4 + c_{t+1} + 2d_{t+1}, \text{ if } c_t, c_{t+1} \text{ both even} \\ c_t = 5 + c_{t+1} + 2d_{t+1}, \text{ if one even one odd} \\ c_t = 6 + c_{t+1} + 2d_{t+1}, \text{ if } c_t, c_{t+1} \text{ both odd} \\ 3 \mid d_t \\ \sum c_t + \sum d_t = n. \end{array} \tag{3.26}$$

From the restrictions in (3.26), we can rewrite matrix (3.25) as:

$$\begin{pmatrix} 4(s-1) + i_s + i_{s-1} + i_{s-2} + \cdots + i_1 + 6d_2 + \cdots + 6d_s & \cdots & 4 + i_s + i_{s-1} + 6d_s & i_s \\ 3d_1 & \cdots & 3d_{s-1} & 3d_s \end{pmatrix}, \tag{3.27}$$

with $i_s \in \{0,1\}$ and $i_t = \begin{cases} 0, & \sum_{j=t}^{s} i_j \equiv \sum_{j=t+1}^{s} i_j \equiv 0 \ (mod\ 2), \\ 1, & \sum_{j=t}^{s} i_j \not\equiv \sum_{j=t+1}^{s} i_j \ (mod\ 2), \\ 2, & \sum_{j=t}^{s} i_j \equiv \sum_{j=t+1}^{s} i_j \equiv 1 \ (mod\ 2). \end{cases}$

We associate a three-line matrix to (3.27) by subtracting i_t from the t-th entry in the first row and putting i_t as the t-th entry of a new third row:

$$\begin{pmatrix} 4(s-1) + i_s + i_{s-1} + i_{s-2} + \cdots + i_2 + 3d_2 + \cdots + 3d_s & \cdots & 4 + i_s + 3d_s & 0 \\ 3d_1 & \cdots & 3d_{s-1} & 3d_s \\ i_1 & \cdots & i_{s-1} & i_s \end{pmatrix}. \tag{3.28}$$

Then, in order to obtain a type $\mathcal{R}$ plane partition, we add 1 to each entry of this matrix, obtaining:

$$\begin{pmatrix} 4s - 3 + i_s + i_{s-1} + i_{s-2} + \cdots + i_2 + 3d_2 + \cdots + 3d_s & \cdots & s + i_s + 3d_s & 1 \\ 1 + 3d_1 & \cdots & 1 + 3d_{s-1} & 1 + 3d_s \\ 1 + i_1 & \cdots & 1 + i_{s-1} & 1 + i_s \end{pmatrix}. \tag{3.29}$$

From the matrix representation (3.29), the next theorem presents the result for the mock theta function $\chi(q)$ in terms of plane partitions.

Theorem 3.7. *The mock theta function $\rho(q)$ is the generating function for type $\mathcal{R}$ plane partitions having weight $(-1)^{\frac{1}{3}(-s+(\sum_{t=2}^{s+1} y_t))}$ and satisfying:*

i. $\lambda_1 = 1, 1 \leq \lambda_j - \lambda_{j-1} \leq 3, 1 \leq \lambda_{s-1} - \lambda_s \leq 2,$

ii. $y_1 = 1, y_j \equiv 1 (mod\ 3),$

iii. $x_{s+1} = 1, x_s = 1, x_j - x_{j+1} = 2 + \lambda_{j+2} - \lambda_{j+1} + y_{j+2}.$

Proof. Recall the preceding discussion Recall the interpretation for $\chi(q)$ as the generating functions for the set of two-line matrices in the form (3.25) with restrictions (3.26). As a preceding building, the three-line matrix representation is given by (3.29), namely,

$$\begin{pmatrix} 4s-3+i_s+i_{s-1}+i_{s-2}+\cdots+i_2+3d_2+\cdots+3d_s & \cdots & s+i_s+3d_s & 1 \\ 1+3d_1 & \cdots & 1+3d_{s-1} & 1+3d_s \\ 1+i_1 & \cdots & 1+i_{s-1} & 1+i_s \end{pmatrix}.$$

The plane partitions obtained from (3.29) have the parameters $y_1 = 1$, $y_j = 1+3d_{j-1}$, then $y_j \equiv 1 (\text{mod } 3)$, for $j = 2, \ldots, s+1$, $\lambda_1 = 1$, $\lambda_{j+1} - \lambda_j = 1+i_j$, since $i_s \in \{0, 1\}$ then $1 \leq \lambda_s - \lambda_{s-1} \leq 2$, and $1 \leq \lambda_j - \lambda_{j-1} \leq 3$, for $j = 1, \ldots, s-1$. Also,

$$\begin{aligned} x_1 &= 4s-3+i_s+i_{s-1}+i_{s-2}+\cdots+i_2+3d_2+\cdots+3d_s, \\ &\vdots \\ x_s &= s+i_s+3d_s, \\ x_{s+1} &= 1. \end{aligned}$$

Then, we have $x_j - x_{j+1} = i_j + 3d_j = \lambda_{j+1} - \lambda_j + y_{j+1} - 2$. Finally, the weight of the plane partitions are:

$$(-1)^{\frac{1}{3}\sum_{t=1}^{s} d_t} = (-1)^{\frac{1}{3}(-s+(\sum_{t=2}^{s+1} y_t))},$$

since $d_j = y_{j+1} - 1$, for $j = 1, \ldots, s$.

Conversely, it is easy to see that, given a type $\mathcal{R}$ plane partition satisfying conditions *i*, *ii*, and *iii*, we can associate to it a unique matrix of the form (3.20). □

Consider, for example, the third-order mock theta function:

$$f_0(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n},$$

where $(a;q)_0 = 1$ and $(a;q)_n = \prod_{k=0}^{n-1}(1 - aq^k)$ for any positive integer n. In [7] this mock theta function was shown to be the generating function for two-line matrices of the form

$$\begin{pmatrix} c_1 & c_2 & c_3 & \cdots & c_s \\ d_1 & d_2 & d_3 & \cdots & d_s \end{pmatrix} \tag{3.30}$$

with non-negative integer entries satisfying

$$\begin{aligned} &c_s = 1 \\ &c_t = 2 + c_{t+1} + d_{t+1} \\ &\sum c_t + \sum d_t = n \end{aligned} \tag{3.31}$$

and weight $(-1)^{\sum_{t=1}^{s} d_t}$. From the restrictions in (3.31), we can rewrite matrix (3.30) as:

$$\begin{pmatrix} (2s-1)+d_2+\cdots+d_s & \cdots & 3+d_s & 1 \\ d_1 & \cdots & d_{s-1} & d_s \end{pmatrix}. \tag{3.32}$$

We associate a three-line matrix to (3.32) by putting 0 as the t-th entry of a new third row:

$$\begin{pmatrix} (2s-1)+d_2+\cdots+d_s & \cdots & 3+d_s & 1 \\ d_1 & \cdots & d_{s-1} & d_s \\ 0 & \cdots & 0 & 0 \end{pmatrix}. \tag{3.33}$$

Note that matrix (3.33) may have zero entries. Then, in order to obtain a type $\mathcal{R}$ plane partition, we add 1 to each entry of this matrix, obtaining:

$$\begin{pmatrix} 2s+d_2+\cdots+d_s & \cdots & 4+d_s & 2 \\ 1+d_1 & \cdots & 1+d_{s-1} & 1+d_s \\ 1 & \cdots & 1 & 1 \end{pmatrix}. \tag{3.34}$$

This procedure can be easily reverted to get (3.30) again.

Now we can use the procedure described in the last subsection to associate a unique type $\mathcal{R}$ plane partition to each matrix (3.34). The next theorem presents the result for the mock theta function $f_0(q)$. In this theorem, we use the parameters from Figure 4 to describe the plane partitions.

Theorem 3.8. *The mock theta function $f_0(q)$ is the generating function for type $\mathcal{R}$ plane partitions having weight $(-1)^{(-s+(\sum_{t=2}^{s+1} y_t))}$ and satisfying:*

i. $\lambda_1 = 1,\ \lambda_j - \lambda_{j-1} = 1,$

ii. $y_1 = 1, y_j \geq 1,$

iii. $x_{s+1} = 1, x_s = 2, x_j - x_{j+1} = 1 + y_{j+2}.$

Proof. □

Consider, for example, the third-order mock theta function:

$$F_0(q) = \sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q;q^2)_n},$$

where $(a;q)_0 = 1$ and $(a;q)_n = \prod_{k=0}^{n-1}(1-aq^k)$ for any positive integer n. In [7] this mock theta function was shown to be the generating function for two-line matrices of the form

$$\begin{pmatrix} c_1 & c_2 & c_3 & \cdots & c_s \\ d_1 & d_2 & d_3 & \cdots & d_s \end{pmatrix} \tag{3.35}$$

with non-negative integer entries satisfying

$$\begin{array}{l} c_s = 2 \\ c_t = 4 + c_{t+1} + d_{t+1} \\ \sum c_t + \sum d_t = n \end{array} \tag{3.36}$$

From the restrictions in (3.36), we can rewrite matrix (3.35) as:

$$\begin{pmatrix} 2(2s-1) + d_2 + \cdots + d_s & \cdots & 6 + d_s & 2 \\ d_1 & \cdots & d_{s-1} & d_s \end{pmatrix}. \tag{3.37}$$

We associate a three-line matrix to (3.37) by putting 0 as the t-th entry of a new third row:

$$\begin{pmatrix} 2(2s-1) + d_2 + \cdots + d_s & \cdots & 6 + d_s & 2 \\ d_1 & \cdots & d_{s-1} & d_s \\ 0 & \cdots & 0 & 0 \end{pmatrix}. \tag{3.38}$$

Note that matrix (3.33) may have zero entries. Then, in order to obtain a type $\mathcal{R}$ plane partition, we add 1 to each entry of this matrix, obtaining:

$$\begin{pmatrix} 4s - 1 + d_2 + \cdots + d_s & \cdots & 7 + d_s & 3 \\ 1 + d_1 & \cdots & 1 + d_{s-1} & 1 + d_s \\ 1 & \cdots & 1 & 1 \end{pmatrix}. \tag{3.39}$$

This procedure can be easily reverted to get (3.35) again.

Now we can use the procedure described in the last subsection to associate a unique type $\mathcal{R}$ plane partition to each matrix (3.39). The next theorem presents the result for the mock theta function $F_0(q)$. In this theorem, we use the parameters from Figure 4 to describe the plane partitions.

Theorem 3.9. *The mock theta function $F_0(q)$ is the generating function for type $\mathcal{R}$ plane partitions satisfying:*

i. $\lambda_1 = 1,\ \lambda_j - \lambda_{j-1} = 1,$

ii. $y_1 = 1, y_j \geq 1,$

iii. $x_{s+1} = 1, x_s = 3, x_j - x_{j+1} = y_{j+2} + 3.$

Proof. □

Consider, for example, the third-order mock theta function:

$$\Psi_0(q) = 1 + \frac{1}{2}\sum_{n=1}^{\infty}(-1;q)_n q^{\binom{n+1}{2}},$$

where $(a;q)_0 = 1$ and $(a;q)_n = \prod_{k=0}^{n-1}(1-aq^k)$ for any positive integer n. In [7] this mock theta function was shown to be the generating function for two-line matrices of the form

$$\begin{pmatrix} c_1 & c_2 & c_3 & \cdots & c_s \\ d_1 & d_2 & d_3 & \cdots & d_s \end{pmatrix} \tag{3.40}$$

with non-negative integer entries satisfying

$$\begin{aligned} & c_s = 1 \\ & d_s = 0 \\ & c_t = 1 + c_{t+1} + d_{t+1} \\ & d_t \in \{0,1\} \\ & \sum c_t + \sum d_t = n \end{aligned} \tag{3.41}$$

From the restrictions in (3.41), we can rewrite matrix (3.40) as:

$$\begin{pmatrix} s + d_2 + \cdots d_{s-1} & \cdots & 2 & 1 \\ d_1 & \cdots & d_{s-1} & 0 \end{pmatrix}. \tag{3.42}$$

We associate a three-line matrix to (3.42) by putting 0 as the t-th entry of a new third row:

$$\begin{pmatrix} s + d_2 + \cdots + d_{s-1} & \cdots & 2 & 1 \\ d_1 & \cdots & d_{s-1} & 0 \\ 0 & \cdots & 0 & 0 \end{pmatrix}. \tag{3.43}$$

Note that matrix (3.43) may have zero entries. Then, in order to obtain a type $\mathcal{R}$ plane partition, we add 1 to each entry of this matrix, obtaining:

$$\begin{pmatrix} 1 + s + d_2 + \cdots + d_{s_1} & \cdots & 3 & 2 \\ 1 + d_1 & \cdots & 1 + d_{s-1} & 1 \\ 1 & \cdots & 1 & 1 \end{pmatrix}. \tag{3.44}$$

This procedure can be easily reverted to get (3.40) again.

Now we can use the procedure described in the last subsection to associate a unique type $\mathcal{R}$ plane partition to each matrix (3.44). The next theorem presents the result for the mock theta function $\Psi_0(q)$. In this theorem, we use the parameters from Figure 4 to describe the plane partitions.

Theorem 3.10. *The mock theta function $\Psi_0(q)$ is the generating function for type $\mathcal{R}$ plane partitions satisfying:*

i. $\lambda_1 = 1,\ \lambda_j - \lambda_{j-1} = 1,$

ii. $y_1 = 1, y_{s+1} = 1, 0 \leq |y_j - y_{j+1}| \leq 1,$

iii. $x_{s+1} = 1, x_s = 2, x_{s-1} = 3, x_j - x_{j+1} = y_{j+2}.$

Proof. □

Consider, for example, the third-order mock theta function:

$$\Phi_0(q) = \sum_{n=1}^{\infty}(-q;q^2)_n q^{n^2},$$

where $(a;q)_0 = 1$ and $(a;q)_n = \prod_{k=0}^{n-1}(1-aq^k)$ for any positive integer n. In [7] this mock theta function was shown to be the generating function for two-line matrices of the form

$$\begin{pmatrix} c_1 & c_2 & c_3 & \cdots & c_s \\ d_1 & d_2 & d_3 & \cdots & d_s \end{pmatrix} \tag{3.45}$$

with non-negative integer entries satisfying

$$\begin{aligned} & c_s \in \{1,2\} \\ & c_t = 2 + c_{t+1} + 2d_{t+1} \\ & d_t \in \{0,1\} \\ & \sum c_t + \sum d_t = n \end{aligned} \tag{3.46}$$

From the restrictions in (3.46), we can rewrite matrix (3.45) as:

$$\begin{pmatrix} 2s + c_s + 2d_2 + \cdots 2d_s & \cdots & 2 + c_s + 2d_s & c_s \\ d_1 & \cdots & d_{s-1} & d_s \end{pmatrix}. \tag{3.47}$$

We associate a three-line matrix to (3.47) by putting 0 as the t-th entry of a new third row:

$$\begin{pmatrix} 2s + c_s + 2d_2 + \cdots 2d_s & \cdots & 2 + c_s + 2d_s & c_s \\ d_1 & \cdots & d_{s-1} & d_s \\ 0 & \cdots & 0 & 0 \end{pmatrix}. \tag{3.48}$$

Note that matrix (3.48) may have zero entries. Then, in order to obtain a type $\mathcal{R}$ plane partition, we add 1 to each entry of this matrix, obtaining:

$$\begin{pmatrix} (2s+1)+c_s+2d_2+\cdots 2d_s & \cdots & 3+c_s+2d_s & 1+c_s \\ 1+d_1 & \cdots & 1+d_{s-1} & 1+d_s \\ 1 & \cdots & 1 & 1 \end{pmatrix}. \tag{3.49}$$

This procedure can be easily reverted to get (3.45) again.

Now we can use the procedure described in the last subsection to associate a unique type $\mathcal{R}$ plane partition to each matrix (3.49). The next theorem presents the result for the mock theta function $\Phi_0(q)$. In this theorem, we use the parameters from Figure 4 to describe the plane partitions.

Theorem 3.11. *The mock theta function $\Phi_0(q)$ is the generating function for type $\mathcal{R}$ plane partitions satisfying:*

i. $\lambda_1 = 1$, $\lambda_j - \lambda_{j-1} = 1$,

ii. $y_1 = 1, 0| \leq y_j - y_{j+1}| \leq 1$,

iii. $x_{s+1} = 1, 2 \leq x_s \leq 3, x_j - x_{j+1} = 2(y_{j+2})$.

Proof. □

Consider, for example, the third-order mock theta function:

$$f_1(q) = \sum_{n=1}^{\infty} \frac{q^{n^2+n}}{(-q;q)_n},$$

where $(a;q)_0 = 1$ and $(a;q)_n = \prod_{k=0}^{n-1}(1-aq^k)$ for any positive integer n. In [7] this mock theta function was shown to be the generating function for two-line matrices of the form

$$\begin{pmatrix} c_1 & c_2 & c_3 & \cdots & c_s \\ d_1 & d_2 & d_3 & \cdots & d_s \end{pmatrix} \tag{3.50}$$

with non-negative integer entries satisfying

$$\begin{aligned} & c_s = 2 \\ & c_t = 2 + c_{t+1} + d_{t+1} \\ & \sum c_t + \sum d_t = n \end{aligned} \tag{3.51}$$

and weight $(-1)^{\sum_{t=1}^{s} d_t}$.From the restrictions in (3.51), we can rewrite matrix (3.50) as:

$$\begin{pmatrix} 2s + d_2 + \cdots d_s & \cdots & 4 + d_s & 2 \\ d_1 & \cdots & d_{s-1} & d_s \end{pmatrix}. \tag{3.52}$$

We associate a three-line matrix to (3.52) by putting 0 as the t-th entry of a new third row:

$$\begin{pmatrix} 2s + d_2 + \cdots d_s & \cdots & 4 + d_s & 2 \\ d_1 & \cdots & d_{s-1} & d_s \\ 0 & \cdots & 0 & 0 \end{pmatrix}. \tag{3.53}$$

Note that matrix (3.53) may have zero entries. Then, in order to obtain a type $\mathcal{R}$ plane partition, we add 1 to each entry of this matrix, obtaining:

$$\begin{pmatrix} 1 + 2s + d_2 + \cdots d_s & \cdots & 5 + d_s & 3 \\ 1 + d_1 & \cdots & 1 + d_{s-1} & 1 + d_s \\ 1 & \cdots & 1 & 1 \end{pmatrix}. \tag{3.54}$$

This procedure can be easily reverted to get (3.50) again.

Now we can use the procedure described in the last subsection to associate a unique type $\mathcal{R}$ plane partition to each matrix (3.54). The next theorem presents the result for the mock theta function $f_1(q)$. In this theorem, we use the parameters from Figure 4 to describe the plane partitions.

Theorem 3.12. *The mock theta function $f_1(q)$ is the generating function for type $\mathcal{R}$ plane partitions having weight $(-1)^{(-s+(\sum_{t=2}^{s+1} y_t))}$ and satisfying:*

i. $\lambda_1 = 1,\ \lambda_j - \lambda_{j-1} = 1,$

ii. $y_1 = 1, y_j \geq 1,$

iii. $x_{s+1} = 1, x_s = 3, x_j - x_{j+1} = y_{j+2} + 1.$

Proof. □

Consider, for example, the third-order mock theta function:

$$F_1(q) = \sum_{n=1}^{\infty} \frac{q^{2n^2+2n}}{(q;q^2)_{n+1}},$$

where $(a;q)_0 = 1$ and $(a;q)_n = \prod_{k=0}^{n-1}(1-aq^k)$ for any positive integer n. In [7] this mock theta function was shown to be the generating function for two-line matrices of the form

$$\begin{pmatrix} c_1 & c_2 & c_3 & \cdots & c_s \\ d_1 & d_2 & d_3 & \cdots & d_s \end{pmatrix} \tag{3.55}$$

with non-negative integer entries satisfying

$$\begin{array}{l} c_s = 0 \\ c_t = 4 + c_{t+1} + 2d_{t+1} \\ \sum c_t + \sum d_t = n \end{array} \tag{3.56}$$

From the restrictions in (3.56), we can rewrite matrix (3.55) as:

$$\begin{pmatrix} 4s + 2d_2 + \cdots 2d_s & \cdots & 4 + 2d_s & 0 \\ d_1 & \cdots & d_{s-1} & d_s \end{pmatrix}. \tag{3.57}$$

We associate a three-line matrix to (3.57) by putting 0 as the t-th entry of a new third row:

$$\begin{pmatrix} 4s + 2d_2 + \cdots d_s & \cdots & 4 + 2d_s & 0 \\ d_1 & \cdots & d_{s-1} & d_s \\ 0 & \cdots & 0 & 0 \end{pmatrix}. \tag{3.58}$$

Note that matrix (3.58) may have zero entries. Then, in order to obtain a type $\mathcal{R}$ plane partition, we add 1 to each entry of this matrix, obtaining:

$$\begin{pmatrix} 1 + 4s + 2d_2 + \cdots 2d_s & \cdots & 5 + 2d_s & 1 \\ 1 + d_1 & \cdots & 1 + d_{s-1} & 1 + d_s \\ 1 & \cdots & 1 & 1 \end{pmatrix}. \tag{3.59}$$

This procedure can be easily reverted to get (3.55) again.

Now we can use the procedure described in the last subsection to associate a unique type $\mathcal{R}$ plane partition to each matrix (3.59). The next theorem presents the result for the mock theta function $F_1(q)$. In this theorem, we use the parameters from Figure 4 to describe the plane partitions.

Theorem 3.13. *The mock theta function $F_1(q)$ is the generating function for type $\mathcal{R}$ plane partitions satisfying:*

i. $\lambda_1 = 1$, $\lambda_j - \lambda_{j-1} = 1$,

ii. $y_1 = 1, y_j \geq 1$,

iii. $x_{s+1} = 1, x_s = 1, x_j - x_{j+1} = 2 + 2y_{j+2}$.

Proof. □

Consider, for example, the third-order mock theta function:

$$\Psi_1(q) = \sum_{n=1}^{\infty} (-q;q)_n q^{\binom{n+1}{2}},$$

where $(a;q)_0 = 1$ and $(a;q)_n = \prod_{k=0}^{n-1}(1-aq^k)$ for any positive integer n. In [7] this mock theta function was shown to be the generating function for two-line matrices of the form

$$\begin{pmatrix} c_1 & c_2 & c_3 & \cdots & c_s \\ d_1 & d_2 & d_3 & \cdots & d_s \end{pmatrix} \tag{3.60}$$

with non-negative integer entries satisfying

$$\begin{aligned} & c_s = 1 \\ & c_t = 1 + c_{t+1} + d_{t+1} \\ & d_t \in \{0,1\} \\ & \sum c_t + \sum d_t = n \end{aligned} \tag{3.61}$$

From the restrictions in (3.61), we can rewrite matrix (3.60) as:

$$\begin{pmatrix} s + d_2 + \cdots d_{s-1} & \cdots & 2 + d_s & 1 \\ d_1 & \cdots & d_{s-1} & d_s \end{pmatrix}. \tag{3.62}$$

We associate a three-line matrix to (3.62) by putting 0 as the t-th entry of a new third row:

$$\begin{pmatrix} s + d_2 + \cdots + d_{s-1} & \cdots & 2 + d_s & 1 \\ d_1 & \cdots & d_{s-1} & d_s \\ 0 & \cdots & 0 & 0 \end{pmatrix}. \tag{3.63}$$

Note that matrix (3.63) may have zero entries. Then, in order to obtain a type $\mathcal{R}$ plane partition, we add 1 to each entry of this matrix, obtaining:

$$\begin{pmatrix} 1 + s + d_2 + \cdots + d_{s_1} & \cdots & 3 + d_s & 2 \\ 1 + d_1 & \cdots & 1 + d_{s-1} & 1 + d_s \\ 1 & \cdots & 1 & 1 \end{pmatrix}. \tag{3.64}$$

This procedure can be easily reverted to get (3.60) again.

Now we can use the procedure described in the last subsection to associate a unique type $\mathcal{R}$ plane partition to each matrix (3.64). The next theorem presents the result for the mock theta function $\Psi_1(q)$. In this theorem, we use the parameters from Figure 4 to describe the plane partitions.

Theorem 3.14. *The mock theta function $\Psi_1(q)$ is the generating function for type $\mathcal{R}$ plane partitions satisfying:*

i. $\lambda_1 = 1,\ \lambda_j - \lambda_{j-1} = 1,$

ii. $y_1 = 1, 0| \leq y_j - y_{j+1}| \leq 1,$

iii. $x_{s+1} = 1, x_s = 2, x_j - x_{j+1} = y_{j+2}.$

Proof. □

Consider, for example, the third-order mock theta function:

$$\Phi_1(q) = \sum_{n=1}^{\infty} (-q; q^2)_n q^{(n+1)^2},$$

where $(a;q)_0 = 1$ and $(a;q)_n = \prod_{k=0}^{n-1}(1 - aq^k)$ for any positive integer n. In [7] this mock theta function was shown to be the generating function for two-line matrices of the form

$$\begin{pmatrix} c_1 & c_2 & c_3 & \cdots & c_s \\ d_1 & d_2 & d_3 & \cdots & d_s \end{pmatrix} \tag{3.65}$$

with non-negative integer entries satisfying

$$\begin{aligned} & c_s = 1 \\ & d_s = 0 \\ & c_t = 2 + c_{t+1} + 2d_{t+1} \\ & d_t \in \{0, 1\} \\ & \sum c_t + \sum d_t = n \end{aligned} \tag{3.66}$$

From the restrictions in (3.66), we can rewrite matrix (3.65) as:

$$\begin{pmatrix} 2s + 1 + 2d_2 + \cdots 2d_{s-1} & \cdots & 3 & 1 \\ d_1 & \cdots & d_{s-1} & 0 \end{pmatrix}. \tag{3.67}$$

We associate a three-line matrix to (3.67) by putting 0 as the t-th entry of a new third row:

$$\begin{pmatrix} 2s + 1 + 2d_2 + \cdots 2d_{s-1} & \cdots & 3 & 1 \\ d_1 & \cdots & d_{s-1} & 0 \\ 0 & \cdots & 0 & 0 \end{pmatrix}. \tag{3.68}$$

Note that matrix (3.68) may have zero entries. Then, in order to obtain a type $\mathcal{R}$ plane partition, we add 1 to each entry of this matrix, obtaining:

$$\begin{pmatrix} (2s+2)+2d_2+\cdots 2d_{s-1} & \cdots & 4 & 2 \\ 1+d_1 & \cdots & 1+d_{s-1} & 1 \\ 1 & \cdots & 1 & 1 \end{pmatrix}. \tag{3.69}$$

This procedure can be easily reverted to get (3.65) again.

Now we can use the procedure described in the last subsection to associate a unique type $\mathcal{R}$ plane partition to each matrix (3.69). The next theorem presents the result for the mock theta function $\Phi_1(q)$. In this theorem, we use the parameters from Figure 4 to describe the plane partitions.

Theorem 3.15. *The mock theta function $\Phi_1(q)$ is the generating function for type $\mathcal{R}$ plane partitions satisfying:*

i. $\lambda_1 = 1,\ \lambda_j - \lambda_{j-1} = 1,$

ii. $y_1 = y_{s+1} = 1, 0 \leq |y_j - y_{j+1}| \leq 1,$

iii. $x_{s+1} = 1, x_s = 2, x_{s-1} = 4, x_j - x_{j+1} = 2y_{j+2}.$

Proof. □

Consider, for example, the third-order mock theta function:

$$\Phi(q) = \sum_{n=0}^{\infty} \frac{(-1)^n (q;q^2)_n q^{n^2}}{(-q;q)_{2n}},$$

where $(a;q)_0 = 1$ and $(a;q)_n = \prod_{k=0}^{n-1}(1-aq^k)$ for any positive integer n. In [7] this mock theta function was shown to be the generating function for two-line matrices of the form

$$\begin{pmatrix} c_1 & c_2 & c_3 & \cdots & c_s \\ d_1 & d_2 & d_3 & \cdots & d_s \end{pmatrix} \tag{3.70}$$

with non-negative integer entries satisfying

$$\begin{aligned} & c_s = 0 \text{ and } s \text{ even} \\ & c_t = i_t + c_{t+1} + d_{t+1} \text{ where } \begin{cases} i_t \in \{1,2\}, & \text{if } t \text{ is odd} \\ i_t = 0, & \text{if } t \text{ is even,} \end{cases} \\ & \sum c_t + \sum d_t = n \end{aligned} \tag{3.71}$$

and weight $(-1)^{c_1+d_1}$. From the restrictions in (3.71), we can rewrite matrix (3.70) as:

$$\begin{pmatrix} i_1 + i_2 + i_3 + \cdots i_{s-1} + d_2 + \cdots d_s & \cdots & i_{s-1} + d_s & 0 \\ d_1 & \cdots & d_{s-1} & d_s \end{pmatrix}. \tag{3.72}$$

We associate a three-line matrix to (3.72) by subtracting i_t from the t-th entry in the first row and putting i_t as the t-th entry of a new third row:

$$\begin{pmatrix} i_2 + \cdots + i_{s-1} + d_2 + \cdots + d_s & \cdots & d_s & 0 \\ d_1 & \cdots & d_{s-1} & d_s \\ i_1 & \cdots & i_{s-1} & 0 \end{pmatrix}. \tag{3.73}$$

Note that matrix (3.73) may have zero entries. Then, in order to obtain a type $\mathcal{R}$ plane partition, we add 1 to each entry of this matrix, obtaining:

$$\begin{pmatrix} 1 + i_2 + \cdots + i_{s-1} + d_2 + \cdots + d_s & \cdots & 1 + d_s & 1 \\ 1 + d_1 & \cdots & 1 + d_{s-1} & 1 + d_s \\ 1 + i_1 & \cdots & 1 + i_{s-1} & 1 \end{pmatrix}. \tag{3.74}$$

This procedure can be easily reverted to get (3.74) again.

Now we can use the procedure described in the last subsection to associate a unique type $\mathcal{R}$ plane partition to each matrix (3.74). The next theorem presents the result for the mock theta function $\Phi(q)$. In this theorem, we use the parameters from Figure 4 to describe the plane partitions.

Theorem 3.16. *The mock theta function $\Phi(q)$ is the generating function for type $\mathcal{R}$ plane partitions having weight $(-1)^{x_1+y_2}$ satisfying:*

i. an even number of different parts,

ii. $\lambda_1 = 1, \lambda_{s+1} - \lambda_s = 1, 1 \leq \lambda_j - \lambda_{j-1} \leq 3,$

iii. $y_1 = 1, y_j \geq 1,$

iv. $x_{s+1} = 1, x_s = 1, x_j - x_{j+1} = \lambda_{j+2} - \lambda_{j+1} + y_{j+2} - 2.$

Proof. □

Consider, for example, the third-order mock theta function:

$$\Psi(q) = \sum_{n=0}^{\infty} \frac{(-1)^n (q;q^2)_n q^{(n+1)^2}}{(-q;q)_{2n+1}},$$

where $(a;q)_0 = 1$ and $(a;q)_n = \prod_{k=0}^{n-1}(1-aq^k)$ for any positive integer n. In [7] this mock theta function was shown to be the generating function for two-line matrices of the form

$$\begin{pmatrix} c_1 & c_2 & c_3 & \cdots & c_s \\ d_1 & d_2 & d_3 & \cdots & d_s \end{pmatrix} \tag{3.75}$$

with non-negative integer entries satisfying

$$\begin{aligned} & c_s = 1 \text{ and } s \text{ odd} \\ & c_t = i_t + c_{t+1} + d_{t+1} \text{ where } \begin{cases} i_t \in \{1,2\}, & \text{if } t \text{ odd} \\ i_t = 0, & \text{otherwise,} \end{cases} \\ & \sum c_t + \sum d_t = n \end{aligned} \tag{3.76}$$

and weight $(-1)^{c_1+d_1-1}$, if $s \neq 1$, $(-1)^{1+d_1}$, if $s = 1$,. From the restrictions in (3.76), we can rewrite matrix (3.75) as:

$$\begin{pmatrix} i_1 + i_2 + i_3 + \cdots i_{s-1} + d_2 + \cdots d_s & \cdots & 1 + i_{s-1} + d_s & 1 \\ d_1 & \cdots & d_{s-1} & d_s \end{pmatrix}. \tag{3.77}$$

We associate a three-line matrix to (3.77) by subtracting i_t from the t-th entry in the first row and putting i_t as the t-th entry of a new third row:

$$\begin{pmatrix} 1 + i_2 + \cdots + i_{s-1} + d_2 + \cdots + d_s & \cdots & 1 + d_s & 1 \\ d_1 & \cdots & d_{s-1} & d_s \\ i_1 & \cdots & i_{s-1} & 0 \end{pmatrix}. \tag{3.78}$$

Note that matrix (3.78) may have zero entries. Then, in order to obtain a type $\mathcal{R}$ plane partition, we add 1 to each entry of this matrix, obtaining:

$$\begin{pmatrix} 2 + i_2 + \cdots + i_{s-1} + d_2 + \cdots + d_s & \cdots & 2 + d_s & 2 \\ 1 + d_1 & \cdots & 1 + d_{s-1} & 1 + d_s \\ 1 + i_1 & \cdots & 1 + i_{s-1} & 1 \end{pmatrix}. \tag{3.79}$$

This procedure can be easily reverted to get (3.79) again.

Now we can use the procedure described in the last subsection to associate a unique type $\mathcal{R}$ plane partition to each matrix (3.79). The next theorem presents the result for the mock theta function $\Psi(q)$. In this theorem, we use the parameters from Figure 4 to describe the plane partitions.

Theorem 3.17. *The mock theta function $\Psi(q)$ is the generating function for type $\mathcal{R}$ plane partitions having weight $(-1)^{x_1+\lambda_2+y_2}$, if $s \neq 1$, $(-1)^{y_2}$, if $s = 1$, and satisfying:*

i. an odd number of different parts,

ii. $\lambda_1 = 1, \lambda_{s+1} - \lambda_s = 1, 1 \leq \lambda_j - \lambda_{j-1} \leq 3,$

iii. $y_1 = 1, y_j \geq 1,$

iv. $x_{s+1} = 1, x_s = 2, x_j - x_{j+1} = \lambda_{j+2} - \lambda_{j+1} + y_{j+2} - 2.$

Proof. □

Consider, for example, the third-order mock theta function:

$$\rho(q) = \sum_{n=0}^{\infty} \frac{(-1)^n(-q;q)_n q^{\binom{n+1}{2}}}{(q;q^2)_{n+1}},$$

where $(a;q)_0 = 1$ and $(a;q)_n = \prod_{k=0}^{n-1}(1-aq^k)$ for any positive integer n. In [7] this mock theta function was shown to be the generating function for two-line matrices of the form

$$\begin{pmatrix} c_1 & c_2 & c_3 & \cdots & c_s \\ d_1 & d_2 & d_3 & \cdots & d_s \end{pmatrix} \tag{3.80}$$

with non-negative integer entries satisfying

$$\begin{array}{l} c_s = 0 \\ c_t = i_t + c_{t+1} + 2d_{t+1} \text{ where } \ i_t \in \{1,2\}, \\ \sum c_t + \sum d_t = n \end{array} \tag{3.81}$$

From the restrictions in (3.81), we can rewrite matrix (3.80) as:

$$\begin{pmatrix} i_1 + i_2 + i_3 + \cdots i_{s-1} + 2d_2 + \cdots 2d_s & \cdots & i_{s-1} + 2d_s & 0 \\ d_1 & \cdots & d_{s-1} & d_s \end{pmatrix}. \tag{3.82}$$

We associate a three-line matrix to (3.82) by subtracting i_t from the t-th entry in the first row and putting i_t as the t-th entry of a new third row:

$$\begin{pmatrix} i_2 + \cdots + i_{s-1} + 2d_2 + \cdots + 2d_s & \cdots & 2d_s & 0 \\ d_1 & \cdots & d_{s-1} & d_s \\ i_1 & \cdots & i_{s-1} & 0 \end{pmatrix}. \tag{3.83}$$

Note that matrix (3.83) may have zero entries. Then, in order to obtain a type $\mathcal{R}$ plane partition, we add 1 to each entry of this matrix, obtaining:

$$\begin{pmatrix} 1 + i_2 + \cdots + i_{s-1} + 2d_2 + \cdots + 2d_s & \cdots & 1 + 2d_s & 1 \\ 1 + d_1 & \cdots & 1 + d_{s-1} & 1 + d_s \\ 1 + i_1 & \cdots & 1 + i_{s-1} & 1 \end{pmatrix}. \tag{3.84}$$

This procedure can be easily reverted to get (3.84) again.

Now we can use the procedure described in the last subsection to associate a unique type $\mathcal{R}$ plane partition to each matrix (3.84). The next theorem presents the result for the mock theta function $\rho(q)$. In this theorem, we use the parameters from Figure 4 to describe the plane partitions.

Theorem 3.18. *The mock theta function $\rho(q)$ is the generating function for type $\mathcal{R}$ plane partitions satisfying:*

i. $\lambda_1 = 1, \lambda_{s+1} - \lambda_s = 1, 2 \leq \lambda_j - \lambda_{j-1} \leq 3,$

ii. $y_1 = 1, y_j \geq 1,$

iii. $x_{s+1} = 1, x_s = 1, x_j - x_{j+1} = \lambda_{j+2} - \lambda_{j+1} + 2y_{j+2} - 3.$

Proof. □

Consider, for example, the third-order mock theta function:

$$\sigma(q) = \sum_{n=0}^{\infty} \frac{(-1)^n(-q;q)_n q^{\binom{n+2}{2}}}{(q;q^2)_{n+1}},$$

where $(a;q)_0 = 1$ and $(a;q)_n = \prod_{k=0}^{n-1}(1 - aq^k)$ for any positive integer n. In [7] this mock theta function was shown to be the generating function for two-line matrices of the form

$$\begin{pmatrix} c_1 & c_2 & c_3 & \cdots & c_s \\ d_1 & d_2 & d_3 & \cdots & d_s \end{pmatrix} \tag{3.85}$$

with non-negative integer entries satisfying

$$\begin{aligned} & c_s = 1 \\ & c_t = i_t + c_{t+1} + 2d_{t+1} \text{ where } \ i_t \in \{1, 2\}, \\ & \sum c_t + \sum d_t = n \end{aligned} \tag{3.86}$$

From the restrictions in (3.86), we can rewrite matrix (3.85) as:

$$\begin{pmatrix} 1 + i_1 + i_2 + i_3 + \cdots i_{s-1} + 2d_2 + \cdots 2d_s & \cdots & 1 + i_{s-1} + 2d_s & 1 \\ d_1 & \cdots & d_{s-1} & d_s \end{pmatrix}. \tag{3.87}$$

We associate a three-line matrix to (3.87) by subtracting i_t from the t-th entry in the first row and putting i_t as the t-th entry of a new third row:

$$\begin{pmatrix} 1 + i_2 + \cdots + i_{s-1} + 2d_2 + \cdots + 2d_s & \cdots & 1 + 2d_s & 1 \\ d_1 & \cdots & d_{s-1} & d_s \\ i_1 & \cdots & i_{s-1} & 0 \end{pmatrix}. \tag{3.88}$$

Note that matrix (3.88) may have zero entries. Then, in order to obtain a type $\mathcal{R}$ plane partition, we add 1 to each entry of this matrix, obtaining:

$$\begin{pmatrix} 2+i_2+\cdots+i_{s-1}+2d_2+\cdots+2d_s & \cdots & 2+2d_s & 2 \\ 1+d_1 & \cdots & 1+d_{s-1} & 1+d_s \\ 1+i_1 & \cdots & 1+i_{s-1} & 1 \end{pmatrix}. \tag{3.89}$$

This procedure can be easily reverted to get (3.89) again.

Now we can use the procedure described in the last subsection to associate a unique type $\mathcal{R}$ plane partition to each matrix (3.89). The next theorem presents the result for the mock theta function $\sigma(q)$. In this theorem, we use the parameters from Figure 4 to describe the plane partitions.

Theorem 3.19. *The mock theta function $\sigma(q)$ is the generating function for type $\mathcal{R}$ plane partitions satisfying:*

i. $\lambda_1 = 1, \lambda_{s+1} - \lambda_s = 1, 2 \leq \lambda_j - \lambda_{j-1} \leq 3,$

ii. $y_1 = 1, y_j \geq 1,$

iii. $x_{s+1} = 1, x_s = 2, x_j - x_{j+1} = \lambda_{j+2} - \lambda_{j+1} + 2y_{j+2} - 3.$

Proof. □

Consider, for example, the third-order mock theta function:

$$\lambda(q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^n (q;q^2)_n}{(-q;q)_n},$$

where $(a;q)_0 = 1$ and $(a;q)_n = \prod_{k=0}^{n-1}(1-aq^k)$ for any positive integer n. In [7] this mock theta function was shown to be the generating function for two-line matrices of the form

$$\begin{pmatrix} c_1 & c_2 & c_3 & \cdots & c_s \\ d_1 & d_2 & d_3 & \cdots & d_s \end{pmatrix} \tag{3.90}$$

with non-negative integer entries satisfying

$$\begin{aligned} &c_s = 1+i_s, i_s \geq 0, \\ &c_t = i_t + c_{t+1} + 2d_{t+1} \text{ where } \ i_t \geq 0 \\ &d_t \in \{0,1\} \\ &\sum c_t + \sum d_t = n \end{aligned} \tag{3.91}$$

and weight $(-1)^{s-1+c_1+\sum_{i=1}^{s} d_i}$. From the restrictions in (3.91), we can rewrite matrix (3.122) as:

$$\begin{pmatrix} 1+i_1+i_2+i_3+\cdots i_{s-1}+2d_2+\cdots 2d_s+i_s & \cdots & 1+i_{s-1}+2d_s+i_s & 1+i_s \\ d_1 & \cdots & d_{s-1} & d_s \end{pmatrix}. \quad (3.92)$$

We associate a three-line matrix to (3.92) by subtracting i_t from the t-th entry in the first row and putting i_t as the t-th entry of a new third row:

$$\begin{pmatrix} 1+i_2+\cdots+i_{s-1}+2d_2+\cdots+2d_s+i_s & \cdots & 1+2d_s+i_s & 1 \\ d_1 & \cdots & d_{s-1} & d_s \\ i_1 & \cdots & i_{s-1} & i_s \end{pmatrix}. \quad (3.93)$$

Note that matrix (3.93) may have zero entries. Then, in order to obtain a type $\mathcal{R}$ plane partition, we add 1 to each entry of this matrix, obtaining:

$$\begin{pmatrix} 2+i_2+\cdots+i_{s-1}+2d_2+\cdots+2d_s+i_s & \cdots & 2+2d_s+i_s & 2 \\ 1+d_1 & \cdots & 1+d_{s-1} & 1+d_s \\ 1+i_1 & \cdots & 1+i_{s-1} & 1+i_s \end{pmatrix}. \quad (3.94)$$

This procedure can be easily reverted to get (3.99) again.

Now we can use the procedure described in the last subsection to associate a unique type $\mathcal{R}$ plane partition to each matrix (3.99). The next theorem presents the result for the mock theta function $\lambda(q)$. In this theorem, we use the parameters from Figure 4 to describe the plane partitions.

Theorem 3.20. *The mock theta function $\lambda(q)$ is the generating function for type $\mathcal{R}$ plane partitions having weight $(-1)^{x_1+\lambda_2-\lambda_1+\sum_{t=2}^{s+1} y_t}$, and satisfying*

i. $\lambda_1 = 1, \lambda_j - \lambda_{j-1} \geq 1,$

ii. $y_1 = 1, 0 \leq |y_j - y_{j+1}| \leq 1,$

iii. $x_{s+1} = 1, x_s = 2, x_j - x_{j+1} = \lambda_{j+2} - \lambda_{j+1} + 2y_{j+2} - 3.$

Proof. □

Consider, for example, the third-order mock theta function:

$$\gamma(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}(q;q)_n}{(q^3;q^3)_n},$$

where $(a;q)_0 = 1$ and $(a;q)_n = \prod_{k=0}^{n-1}(1 - aq^k)$ for any positive integer n. In [7] this mock theta function was shown to be the generating function for two-line matrices of the form

$$\begin{pmatrix} c_1 & c_2 & c_3 & \cdots & c_s \\ d_1 & d_2 & d_3 & \cdots & d_s \end{pmatrix} \tag{3.95}$$

with non-negative integer entries satisfying

$$\begin{aligned} & c_s \in \{1,2\} \\ & c_t = i_t + c_{t+1} + d_{t+1} \text{ where } \ i_t \in \{2,3\} \\ & 3 \mid d_t \\ & \sum c_t + \sum d_t = n \end{aligned} \tag{3.96}$$

and weight $(-1)^{1+c_1+\sum_{i=2}^{s} d_i}$. From the restrictions in (3.91), we can rewrite matrix (3.122) as:

$$\begin{pmatrix} i_1 + i_2 + i_3 + \cdots i_{s-1} + 3d_2 + \cdots 3d_s + c_s & \cdots & i_{s-1} + 3d_s + c_s & c_s \\ 3d_1 & \cdots & 3d_{s-1} & 3d_s \end{pmatrix}. \tag{3.97}$$

We associate a three-line matrix to (3.97) by subtracting i_t from the t-th entry in the first row and putting i_t as the t-th entry of a new third row:

$$\begin{pmatrix} i_2 + \cdots + i_{s-1} + 3d_2 + \cdots + 3d_s + c_s & \cdots & 3d_s + c_s & c_s \\ 3d_1 & \cdots & 3d_{s-1} & 3d_s \\ i_1 & \cdots & i_{s-1} & 0 \end{pmatrix}. \tag{3.98}$$

Note that matrix (3.98) may have zero entries. Then, in order to obtain a type $\mathcal{R}$ plane partition, we add 1 to each entry of this matrix, obtaining:

$$\begin{pmatrix} 1 + i_2 + \cdots + i_{s-1} + 3d_2 + \cdots + 3d_s + c_s & \cdots & 1 + 3d_s + c_s & 1 + c_s \\ 1 + 3d_1 & \cdots & 1 + 3d_{s-1} & 1 + 3d_s \\ 1 + i_1 & \cdots & 1 + i_{s-1} & 1 \end{pmatrix}. \tag{3.99}$$

This procedure can be easily reverted to get (3.99) again.

Now we can use the procedure described in the last subsection to associate a unique type $\mathcal{R}$ plane partition to each matrix (3.99). The next theorem presents the result for the mock theta function $\gamma(q)$. In this theorem, we use the parameters from Figure 4 to describe the plane partitions.

Theorem 3.21. *The mock theta function $\gamma(q)$ is the generating function for type $\mathcal{R}$ plane partitions having weight $(-1)^{x_1+\lambda_2-\lambda_1-1+\frac{1}{3}(-(s-1)+\sum_{t=2}^{s} y_{t+1})}$, and satisfying*

i. $\lambda_1 = 1, \lambda_{s+1} - \lambda_s = 1, 2 \leq \lambda_j - \lambda_{j-1} \leq 3,$

ii. $y_1 = 1, y_j \equiv 1 \ (mod\ 3)$,

iii. $x_{s+1} = 1, 2 \leq x_s \leq 3, x_j - x_{j+1} = \lambda_{j+2} - \lambda_{j+1} + 3y_{j+2} - 4$.

Proof. □

Consider, for example, the third-order mock theta function:

$$\mathcal{F}_0(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^{n+1}; q)_n},$$

where $(a; q)_0 = 1$ and $(a; q)_n = \prod_{k=0}^{n-1}(1 - aq^k)$ for any positive integer n. In [7] this mock theta function was shown to be the generating function for two-line matrices of the form

$$\begin{pmatrix} c_1 & c_2 & c_3 & \cdots & c_s \\ d_1 & d_2 & d_3 & \cdots & d_s \end{pmatrix} \tag{3.100}$$

with non-negative integer entries satisfying

$$\begin{array}{l} c_s = 1, \\ (s+1) | d_t, \\ c_t = 2 + c_{t+1} + \frac{d_{t+1}}{s+1}, \\ \sum c_t + \sum d_t = n. \end{array} \tag{3.101}$$

From the restrictions in (3.123), we can rewrite matrix (3.122) as:

$$\begin{pmatrix} 2s - 1 + e_2 + \cdots + e_s & \cdots & 3 + e_s & 1 \\ (s+1)e_1 & \cdots & (s+1)e_{s-1} & (s+1)e_s \end{pmatrix}, \tag{3.102}$$

where $d_t = (s + 1)e_t$. We associate a three-line matrix to (3.124) un the following way:

$$\begin{pmatrix} 2s + e_2 + \cdots + e_s & \cdots & 4 + e_s & 2 \\ 1 + (s+1)e_1 & \cdots & 1 + (s+1)e_{s-1} & 1 + (s+1)e_s \\ 1 & \cdots & 1 & 1 \end{pmatrix}. \tag{3.103}$$

This procedure can be easily reverted to get (3.126) again.

Now we can use the procedure described in the last subsection to associate a unique type $\mathcal{R}$ plane partition to each matrix (3.126). The next theorem presents the result for the mock theta function $\mathcal{F}_0(q)$. In this theorem, we use the parameters from Figure 4 to describe the plane partitions.

Theorem 3.22. *The mock theta function $\mathcal{F}_0(q)$ is the generating function for type $\mathcal{R}$ plane partitions satisfying*

i. $\lambda_1 = 1, \lambda_j - \lambda_{j-1} = 1,$

ii. $y_1 = 1, y_j \equiv 1 (mod\ m)$, *where* m *is the number of different summands,*

iii. $x_{s+1} = 1, x_s = 2, x_j - x_{j+1} = 2 + \frac{y_{j+2}-1}{m}$.

Proof. □

Consider, for example, the third-order mock theta function:

$$\mathcal{F}_1(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^n;q)_n},$$

where $(a;q)_0 = 1$ and $(a;q)_n = \prod_{k=0}^{n-1}(1-aq^k)$ for any positive integer n. In [7] this mock theta function was shown to be the generating function for two-line matrices of the form

$$\begin{pmatrix} c_1 & c_2 & c_3 & \cdots & c_s \\ d_1 & d_2 & d_3 & \cdots & d_s \end{pmatrix} \tag{3.104}$$

with non-negative integer entries satisfying

$$\begin{aligned} & c_s = 1, \\ & s|d_t, \\ & c_t = 2 + c_{t+1} + \tfrac{d_{t+1}}{s}, \\ & \sum c_t + \sum d_t = n. \end{aligned} \tag{3.105}$$

From the restrictions in (3.123), we can rewrite matrix (3.122) as:

$$\begin{pmatrix} 2s-1+e_2+\cdots+e_s & \cdots & 3+e_s & 1 \\ se_1 & \cdots & se_{s-1} & se_s \end{pmatrix}, \tag{3.106}$$

where $d_t = se_t$. We associate a three-line matrix to (3.124) un the following way:

$$\begin{pmatrix} 2s+e_2+\cdots+e_s & \cdots & 4+e_s & 2 \\ 1+se_1 & \cdots & 1+se_{s-1} & 1+se_s \\ 1 & \cdots & 1 & 1 \end{pmatrix}. \tag{3.107}$$

This procedure can be easily reverted to get (3.126) again.

Now we can use the procedure described in the last subsection to associate a unique type $\mathcal{R}$ plane partition to each matrix (3.126). The next theorem presents the result for the mock theta function $\mathcal{F}_1(q)$. In this theorem, we use the parameters from Figure 4 to describe the plane partitions.

Theorem 3.23. *The mock theta function $\mathcal{F}_1(q)$ is the generating function for type $\mathcal{R}$ plane partitions satisfying*

i. $\lambda_1 = 1, \lambda_j - \lambda_{j-1} = 1,$

ii. $y_1 = 1, y_j \equiv 1 (mod\ m-1)$, *where m is the number of different summands,*

iii. $x_{s+1} = 1, x_s = 2, x_j - x_{j+1} = 2 + \frac{y_{j+2}-1}{m-1}.$

Proof. □

Consider, for example, the third-order mock theta function:

$$\mathcal{F}_2(q) = \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q^{n+1}; q)_{n+1}},$$

where $(a; q)_0 = 1$ and $(a; q)_n = \prod_{k=0}^{n-1}(1 - aq^k)$ for any positive integer n. In [7] this mock theta function was shown to be the generating function for two-line matrices of the form

$$\begin{pmatrix} c_1 & c_2 & c_3 & \cdots & c_s \\ d_1 & d_2 & d_3 & \cdots & d_s \end{pmatrix} \tag{3.108}$$

with non-negative integer entries satisfying

$$\begin{aligned} &c_s = 0, \\ &s | d_t, \\ &c_t = 2 + c_{t+1} + \tfrac{d_{t+1}}{s}, \\ &\textstyle\sum c_t + \sum d_t = n. \end{aligned} \tag{3.109}$$

From the restrictions in (3.123), we can rewrite matrix (3.122) as:

$$\begin{pmatrix} 2(s-1) + e_2 + \cdots + e_s & \cdots & 3 + e_s & 0 \\ se_1 & \cdots & se_{s-1} & se_s \end{pmatrix}, \tag{3.110}$$

where $d_t = se_t$. We associate a three-line matrix to (3.124) un the following way:

$$\begin{pmatrix} 2s - 1 + e_2 + \cdots + e_s & \cdots & 3 + e_s & 1 \\ 1 + se_1 & \cdots & 1 + se_{s-1} & 1 + se_s \\ 1 & \cdots & 1 & 1 \end{pmatrix}. \tag{3.111}$$

This procedure can be easily reverted to get (3.126) again.

Now we can use the procedure described in the last subsection to associate a unique type $\mathcal{R}$ plane partition to each matrix (3.126). The next theorem presents the result for the mock theta function $\mathcal{F}_2(q)$. In this theorem, we use the parameters from Figure 4 to describe the plane partitions.

Theorem 3.24. *The mock theta function $\mathcal{F}_2(q)$ is the generating function for type $\mathcal{R}$ plane partitions satisfying*

i. $\lambda_1 = 1, \lambda_j - \lambda_{j-1} = 1,$

ii. $y_1 = 1, y_j \equiv 1 (mod\ m-1)$, *where m is the number of different summands,*

iii. $x_{s+1} = 1 = x_s, x_j - x_{j+1} = 2 + \frac{y_{j+2}-1}{m-1}$.

Proof. □

Consider, for example, the third-order mock theta function:

$$\mathcal{S}_0(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}(-q;q^2)_n}{(-q^2;q^2)_n},$$

where $(a;q)_0 = 1$ and $(a;q)_n = \prod_{k=0}^{n-1}(1-aq^k)$ for any positive integer n. In [7] this mock theta function was shown to be the generating function for two-line matrices of the form

$$\begin{pmatrix} c_1 & c_2 & c_3 & \cdots & c_s \\ d_1 & d_2 & d_3 & \cdots & d_s \end{pmatrix} \tag{3.112}$$

with non-negative integer entries satisfying

$$\begin{aligned} & c_s \in \{1,2\} \\ & 2|d_t, \\ & c_t = \begin{cases} 2 + c_{t+1} + d_{t+1}, & \text{if } c_t \equiv c_{t+1} \equiv 1 (\text{mod } 2) \\ 3 + c_{t+1} + d_{t+1}, & \text{if } c_t \not\equiv c_{t+1} (\text{mod } 2) \\ 4 + c_{t+1} + d_{t+1}, & \text{if } c_t \equiv c_{t+1} \equiv 0 (\text{mod } 2) \end{cases} \\ & \sum c_t + \sum d_t = n, \end{aligned} \tag{3.113}$$

and having height $(-1)^{\frac{1}{2}\sum_{t=1}^{s} d_t}$. We rewrite conditions (3.123) as:

$$\begin{aligned} & c_s = 1 + i_s, i_s \in \{0,1\} \\ & d_t = 2e_t, \\ & c_t = 2 + i_t + c_{t+1} + 2e_{t+1}, t = 1, \ldots, s-1, i_t \in \{0,1,2\}, \\ & \text{where } i_t = \begin{cases} 0, & \text{if } c_t \equiv c_{t+1} \equiv 1 (\text{mod } 2) \\ 1, & \text{if } c_t \not\equiv c_{t+1} (\text{mod } 2) \\ 2, & \text{if } c_t \equiv c_{t+1} \equiv 0 (\text{mod } 2) \end{cases} \\ & \sum c_t + \sum d_t = n, \end{aligned} \tag{3.114}$$

and weight given by $(-1)^{\sum_{t=1}^{s} e_t}$. From the restrictions in (3.114), we can rewrite matrix (3.122) as:

$$\begin{pmatrix} 2s-1+i_1+\cdots+i_{s-1}+i_s+2e_2+\cdots+2e_s & \cdots & 3+i_{s-1}+i_s+2e_s & 1+i_s \\ 2e_1 & \cdots & 2e_{s-1} & 2e_s \end{pmatrix}. \tag{3.115}$$

We associate a three-line matrix to (3.124) un the following way:

$$\begin{pmatrix} 2s+i_2+\cdots+i_{s-1}+i_s+2e_2+\cdots+2e_s & \cdots & 4+i_s+2e_s & 2 \\ 1+2e_1 & \cdots & 1+2e_{s-1} & 1+2e_s \\ 1+i_1 & \cdots & 1+i_{s-1} & 1+i_s \end{pmatrix}. \tag{3.116}$$

This procedure can be easily reverted to get (3.126) again.

Now we can use the procedure described in the last subsection to associate a unique type $\mathcal{R}$ plane partition to each matrix (3.126). The next theorem presents the result for the mock theta function $\mathcal{S}_0(q)$. In this theorem, we use the parameters from Figure 4 to describe the plane partitions.

Theorem 3.25. *The mock theta function $\mathcal{S}_0(q)$ is the generating function for type $\mathcal{R}$ plane partitions having weight $(-1)^{\frac{1}{2}\sum_{j=1}^{s}(y_j-1)}$ and satisfying:*

i. $\lambda_1 = 1, \lambda_{s+1} - \lambda_s \in \{1, 2\}$,

ii. $y_1 = 1, y_j \equiv 1(mod\ 2)$,

iii. $x_{s+1} = 1, x_s = 2, x_j - x_{j+1} = \lambda_{j+2} - \lambda_{j+1} + y_{j+2}$.

iv. $\lambda_{j+1} - \lambda_j = 1 + r_j$, *where*

$$r_j = \begin{cases} 0, & \text{if } x_j + \lambda_{j+1} - \lambda_j \equiv x_{j+1} + \lambda_{j+2} - \lambda_{j+1} \equiv 1(mod\ 2) \\ 1, & \text{if } x_j + \lambda_{j+1} - \lambda_j \not\equiv x_{j+1} + \lambda_{j+2} - \lambda_{j+1}(mod\ 2) \\ 2, & \text{if } x_j + \lambda_{j+1} - \lambda_j \equiv x_{j+1} + \lambda_{j+2} - \lambda_{j+1} \equiv 0(mod\ 2) \end{cases}$$

Proof. □

Consider, for example, the third-order mock theta function:

$$T_0(q) = \sum_{n=0}^{\infty} \frac{q^{(n+2)(n+1)}(-q^2;q^2)_n}{(-q;q^2)_{n+1}},$$

where $(a;q)_0 = 1$ and $(a;q)_n = \prod_{k=0}^{n-1}(1-aq^k)$ for any positive integer n. In [7] this mock theta function was shown to be the generating function for two-line matrices of the form

$$\begin{pmatrix} c_1 & c_2 & c_3 & \cdots & c_s \\ d_1 & d_2 & d_3 & \cdots & d_s \end{pmatrix} \tag{3.117}$$

with non-negative integer entries satisfying

$$\begin{aligned} & c_s = 2 \\ & c_t = i_t + c_{t+1} + 2d_{t+1} \text{ where } \; i_t \in \{2,4\}, \\ & \sum c_t + \sum d_t = n \end{aligned} \tag{3.118}$$

and weight $(-1)^n$. From the restrictions in (3.123), we can rewrite matrix (3.122) as:

$$\begin{pmatrix} i_1 + i_2 + i_3 + \cdots i_{s-1} + 2d_2 + \cdots 2d_s + 2 & \cdots & i_{s-1} + 2d_s + 2 & 2 \\ d_1 & \cdots & d_{s-1} & d_s \end{pmatrix}. \tag{3.119}$$

We associate a three-line matrix to (3.124) by subtracting i_t from the t-th entry in the first row and putting i_t as the t-th entry of a new third row:

$$\begin{pmatrix} i_2 + \cdots + i_{s-1} + 2d_2 + \cdots + 2d_s + 2 & \cdots & 2d_s + 2 & 2 \\ d_1 & \cdots & d_{s-1} & d_s \\ i_1 & \cdots & i_{s-1} & 0 \end{pmatrix}. \tag{3.120}$$

Note that matrix (3.125) may have zero entries. Then, in order to obtain a type $\mathcal{R}$ plane partition, we add 1 to each entry of this matrix, obtaining:

$$\begin{pmatrix} 3 + i_2 + \cdots + i_{s-1} + 2d_2 + \cdots + 2d_s & \cdots & 3 + 2d_s & 3 \\ 1 + d_1 & \cdots & 1 + d_{s-1} & 1 + d_s \\ 1 + i_1 & \cdots & 1 + i_{s-1} & 1 \end{pmatrix}. \tag{3.121}$$

This procedure can be easily reverted to get (3.126) again.

Now we can use the procedure described in the last subsection to associate a unique type $\mathcal{R}$ plane partition to each matrix (3.126). The next theorem presents the result for the mock theta function $T_0(q)$. In this theorem, we use the parameters from Figure 4 to describe the plane partitions.

Theorem 3.26. *The mock theta function $T_0(q)$ is the generating function for type $\mathcal{R}$ plane partitions having weight $(-1)^{-3s-1+\lambda_{s+1}+\sum_{t=1}^{s} x_t + y_{t+1}}$, and satisfying* *i.* $\lambda_1 = 1, \lambda_{s+1} - \lambda_s = 1, 3 \leq \lambda_j - \lambda_{j-1} \leq 5,$

ii. $y_1 = 1, y_j \geq 1,$

iii. $x_{s+1} = 1, x_s = 3, x_j - x_{j+1} = \lambda_{j+2} - \lambda_{j+1} + 2y_{j+2} - 3.$

Proof. □

Consider, for example, the third-order mock theta function:

$$T_1(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}(-q^2;q^2)_n}{(-q;q^2)_{n+1}},$$

where $(a;q)_0 = 1$ and $(a;q)_n = \prod_{k=0}^{n-1}(1-aq^k)$ for any positive integer n. In [7] this mock theta function was shown to be the generating function for two-line matrices of the form

$$\begin{pmatrix} c_1 & c_2 & c_3 & \cdots & c_s \\ d_1 & d_2 & d_3 & \cdots & d_s \end{pmatrix} \tag{3.122}$$

with non-negative integer entries satisfying

$$\begin{array}{l} c_s = 0 \\ c_t = i_t + c_{t+1} + 2d_{t+1} \text{ where } \ i_t \in \{2,4\}, \\ \sum c_t + \sum d_t = n \end{array} \tag{3.123}$$

and weight $(-1)^n$. From the restrictions in (3.123), we can rewrite matrix (3.122) as:

$$\begin{pmatrix} i_1 + i_2 + i_3 + \cdots i_{s-1} + 2d_2 + \cdots 2d_s & \cdots & i_{s-1} + 2d_s & 0 \\ d_1 & \cdots & d_{s-1} & d_s \end{pmatrix}. \tag{3.124}$$

We associate a three-line matrix to (3.124) by subtracting i_t from the t-th entry in the first row and putting i_t as the t-th entry of a new third row:

$$\begin{pmatrix} i_2 + \cdots + i_{s-1} + 2d_2 + \cdots + 2d_s & \cdots & 2d_s & 0 \\ d_1 & \cdots & d_{s-1} & d_s \\ i_1 & \cdots & i_{s-1} & 0 \end{pmatrix}. \tag{3.125}$$

Note that matrix (3.125) may have zero entries. Then, in order to obtain a type $\mathcal{R}$ plane partition, we add 1 to each entry of this matrix, obtaining:

$$\begin{pmatrix} 1 + i_2 + \cdots + i_{s-1} + 2d_2 + \cdots + 2d_s & \cdots & 1 + 2d_s & 1 \\ 1 + d_1 & \cdots & 1 + d_{s-1} & 1 + d_s \\ 1 + i_1 & \cdots & 1 + i_{s-1} & 1 \end{pmatrix}. \tag{3.126}$$

This procedure can be easily reverted to get (3.126) again.

Now we can use the procedure described in the last subsection to associate a unique type $\mathcal{R}$ plane partition to each matrix (3.126). The next theorem presents the result for the mock theta function $T_1(q)$. In this theorem, we use the parameters from Figure 4 to describe the plane partitions.

Theorem 3.27. *The mock theta function $T_1(q)$ is the generating function for type $\mathcal{R}$ plane partitions having weight $(-1)^{-3s-1+\lambda_{s+1}+\sum_{t=1}^{s} x_t+y_{t+1}}$, and satisfying*

i. $\lambda_1 = 1, 3 \leq \lambda_j - \lambda_{j-1} \leq 5$,

ii. $y_1 = 1, y_j \geq 1$,

iii. $x_{s+1} = 1, x_s = 1, x_j - x_{j+1} = \lambda_{j+2} - \lambda_{j+1} + 2y_{j+2} - 3$.

Proof. □

4 Summary of the results

Having described in the last sections how we can obtain representations for the mock theta functions in terms of type $\mathcal{R}$ plane partitions, we summarize the results in the table below. The first column shows the mock theta functions. Column 2 presents the restrictions of the corresponding type $\mathcal{R}$ plane partitions. The last column gives the weight of the plane partitions.

mock theta function	type $\mathcal{R}$ plane partitions	weight
$f(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q\,;q)_n^2}$	$\lambda_1 = 1, \lambda_{j+1} - \lambda_j \geq 1,$ $y_1 = 1, y_j \geq 1,$ $x_{s+1} = 1, x_s = 2,$ $x_j - x_{j+1} = y_{j+2} + \lambda_{j+2} - \lambda_{j+1}$	$(-1)^{y_2 - x_1 - \lambda_2 + \lambda_1}$
$\phi(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q^2\,;q^2)_n}$	$\lambda_1 = 1, \lambda_j - \lambda_{j-1} = 1,$ $y_1 = 1, y_j \equiv 1 (\mathrm{mod}\ 2),$ $x_{s+1} = 1, x_s = 2,$ $x_j - x_{j+1} = 1 + y_{j+1}$	$(-1)^{\frac{1}{2}(-s + (\sum_{t=2}^{s+1} y_t))}$
$\psi(q) = \sum_{n=1}^{\infty} \frac{q^{n^2}}{(q\,;q^2)_n}$	$\lambda_1 = 1, \lambda_j - \lambda_{j-1} = 1,$ $y_1 = 1, y_j \geq 1,$ $x_{s+1} = 1, x_s = 2,$ $x_j - x_{j+1} = 2y_{j+1}$	
$\chi(q) = \sum_{n=0}^{\infty} \frac{(-q\,;q)_n q^{n^2}}{(-q^3\,;q^3)_n}$	$\lambda_1 = 1, 3 \leq \lambda_j - \lambda_{j-1} \leq 4,$ $y_1 = 1, y_j \equiv 1 (\mathrm{mod}\ 3),$ $x_{s+1} = 1, x_s = 1,$ $x_j - x_{j+1} = \lambda_{j+2} - \lambda_{j+1} + y_{j+2} - 2$	$(-1)^{\frac{1}{3}(-s + (\sum_{t=2}^{s+1} y_t))}$
$q\omega(q)$[1] $= \sum_{n=0}^{\infty} \frac{q^{2n(n+1)+1}}{(q\,;q^2)_{n+1}^2}$	$\lambda_1 = 1,$ $\lambda_j - \lambda_{j-1} = 1,$ $y_1 = 1, y_j \geq 1, x_j \equiv 0 (\mathrm{mod}\ 2)$ $x_{s+1} = 1, x_s = 2,$ $x_j - x_{j+1} = 2(y_{j+1} - 1)$	
$\nu(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(-q\,;q^2)_{n+1}}$	$\lambda_1 = 1, \lambda_j - \lambda_{j-1} = 1,$ $y_1 = 1, y_j \geq 1,$ $x_{s+1} = 1,$ $x_s = 1, x_j - x_{j+1} = 2y_{j+2} - 1$	$(-1)^{\sum_{t=1}^{s} x_t + y_{t+1}}$
$\rho(q) = \sum_{n=0}^{\infty} \frac{(q\,;q^2)_{n+1} q^{2n(n+1)}}{(q^3\,;q^6)_{n+1}}$	$\lambda_1 = 1, 1 \leq \lambda_j - \lambda_{j-1} \leq 3, 1 \leq \lambda_{s-1} - \lambda_s \leq 2,$ $y_1 = 1, y_j \equiv 1 (\mathrm{mod}\ 3),$ $x_{s+1} = 1, x_s = 1,$ $x_j - x_{j+1} = 2 + \lambda_{j+2} - \lambda_{j+1} + y_{j+2}$	
$f_0(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q\,;q)_n}$	$\lambda_1 = 1, \lambda_j - \lambda_{j-1} = 1,$ $y_1 = 1, y_j \geq 1,$ $x_{s+1} = 1, x_s = 2,$ $x_j - x_{j+1} = 1 + y_{j+2}$	$(-1)^{(-s + (\sum_{t=2}^{s+1} y_t))}$
$F_0(q) = \sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q\,;q^2)_n}$	$\lambda_1 = 1, \lambda_j - \lambda_{j-1} = 1,$ $y_1 = 1, y_j \geq 1,$ $x_{s+1} = 1, x_s = 3,$ $x_j - x_{j+1} = y_{j+2} + 3$	
$\Psi_0(q) = 1 + \frac{1}{2} \sum_{n=1}^{\infty} (-1\,;q)_n q^{\binom{n+1}{2}}$	$\lambda_1 = 1, \lambda_j - \lambda_{j-1} = 1,$ $y_1 = 1, y_{s+1} = 1, 0 \leq \lvert y_j - y_{j+1} \rvert \leq 1,$ $x_{s+1} = 1, x_s = 2,$ $x_{s-1} = 3, x_j - x_{j+1} = y_{j+2}$	
$\Phi_0(q) = \sum_{n=0}^{\infty} (-q\,;q^2)_n q^{n^2}$	$\lambda_1 = 1, \lambda_j - \lambda_{j-1} = 1,$ $y_1 = 1, 0\lvert \leq y_j - y_{j+1} \rvert \leq 1,$ $x_{s+1} = 1, 2 \leq x_s \leq 3,$ $x_j - x_{j+1} = 2(y_{j+2})$	
$f_1(q) = \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(-q\,;q)_n}$	$\lambda_1 = 1, \lambda_j - \lambda_{j-1} = 1,$ $y_1 = 1, y_j \geq 1,$ $x_{s+1} = 1, x_s = 3,$ $x_j - x_{j+1} = y_{j+2} + 1$	$(-1)^{(-s + (\sum_{t=2}^{s+1} y_t))}$
$F_1(q) = \sum_{n=0}^{\infty} \frac{q^{2n^2+2n}}{(q\,;q^2)_{n+1}}$	$\lambda_1 = 1, \lambda_j - \lambda_{j-1} = 1,$ $y_1 = 1, y_j \geq 1,$ $x_{s+1} = 1, x_s = 1,$ $x_j - x_{j+1} = 2 + 2y_{j+2}$	

[1] See Remark 1 after this table

mock theta function	type $\mathcal{R}$ plane partitions	weight
$\Psi_1(q) = \sum_{n=0}^{\infty} (-q\,;q)_n q^{\binom{n+1}{2}}$	$\lambda_1 = 1, \lambda_j - \lambda_{j-1} = 1,$ $y_1 = 1, 0\vert \leq y_j - y_{j+1}\vert \leq 1,$ $x_{s+1} = 1, x_s = 2,$ $x_j - x_{j+1} = y_{j+2}$	
$\Phi_1(q) = \sum_{n=0}^{\infty} (-q; q^2)_n q^{(n+1)^2}$	$\lambda_1 = 1, \lambda_j - \lambda_{j-1} = 1,$ $y_1 = y_{s+1} = 1, 0 \leq \vert y_j - y_{j+1}\vert \leq 1,$ $x_{s+1} = 1, x_s = 2, x_{s-1} = 4,$ $x_j - x_{j+1} = 2y_{j+2}$	
$\Phi(q) = \sum_{n=0}^{\infty} \frac{(-1)^n (q\,;q^2)_n q^{n^2}}{(-q\,;q)_{2n}}$	an even number of different parts, $\lambda_1 = 1, \lambda_{s+1} - \lambda_s = 1, 1 \leq \lambda_j - \lambda_{j-1} \leq 3,$ $y_1 = 1, y_j \geq 1,$ $x_{s+1} = 1, x_s = 1,$ $x_j - x_{j+1} = \lambda_{j+2} - \lambda_{j+1} + y_{j+2} - 2$	$(-1)^{x_1+y_2}$
$\Psi(q) = \sum_{n=0}^{\infty} \frac{(-1)^n (q\,;q^2)_n q^{(n+1)^2}}{(-q\,;q)_{2n+1}}$	an odd number of different parts, $\lambda_1 = 1, \lambda_{s+1} - \lambda_s = 1, 1 \leq \lambda_j - \lambda_{j-1} \leq 3,$ $y_1 = 1, y_j \geq 1,$ $x_{s+1} = 1, x_s = 2,$ $x_j - x_{j+1} = \lambda_{j+2} - \lambda_{j+1} + y_{j+2} - 2$	$(-1)^{x_1+\lambda_2+y_2}$ if $s \neq 1$; $(-1)^{y_2}$ if $s = 1$
$\rho(q) = \sum_{n=0}^{\infty} \frac{(-q\,;q)_n q^{\binom{n+1}{2}}}{(q\,;q^2)_{n+1}}$	$\lambda_1 = 1, \lambda_{s+1} - \lambda_s = 1, 2 \leq \lambda_j - \lambda_{j-1} \leq 3,$ $y_1 = 1, y_j \geq 1,$ $x_{s+1} = 1, x_s = 1,$ $x_j - x_{j+1} = \lambda_{j+2} - \lambda_{j+1} + 2y_{j+2} - 3$	
$\sigma(q) = \sum_{n=0}^{\infty} \frac{(-q\,;q)_n q^{\binom{n+2}{2}}}{(q\,;q^2)_{n+1}}$	$\lambda_1 = 1, \lambda_{s+1} - \lambda_s = 1, 2 \leq \lambda_j - \lambda_{j-1} \leq 3,$ $y_1 = 1, y_j \geq 1,$ $x_{s+1} = 1, x_s = 2,$ $x_j - x_{j+1} = \lambda_{j+2} - \lambda_{j+1} + 2y_{j+2} - 3$	
$\lambda(q) = \sum_{n=0}^{\infty} \frac{(-1)^n (q\,;q^2)_n q^n}{(-q\,;q)_n}$	$\lambda_1 = 1, \lambda_j - \lambda_{j-1} \geq 1,$ $y_1 = 1, 0 \leq \vert y_j - y_{j+1}\vert \leq 1,$ $x_{s+1} = 1, x_s = 2,$ $x_j - x_{j+1} = \lambda_{j+2} - \lambda_{j+1} + 2y_{j+2} - 3$	$(-1)^{x_1+\lambda_2-\lambda_1+\sum_{t=2}^{s+1} y_t}$
$\gamma(q) = \sum_{n=0}^{\infty} \frac{(q\,;q)_n q^{n^2}}{(q^3\,;q^3)_n}$	$\lambda_1 = 1, \lambda_{s+1} - \lambda_s = 1, 2 \leq \lambda_j - \lambda_{j-1} \leq 3,$ $y_1 = 1, y_j \equiv\ 1 \pmod 3,$ $x_{s+1} = 1, 2 \leq x_s \leq 3,$ $x_j - x_{j+1} = \lambda_{j+2} - \lambda_{j+1} + 3y_{j+2} - 4$	$(-1)^{x_1+\lambda_2+\frac{1}{3}(-(s-1)+\sum_{t=2}^{s} y_{t+1})}$
$\mathcal{F}_0(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^{n+1}\,;q)_n}$	$\lambda_1 = 1, \lambda_j - \lambda_{j-1} = 1,$ $y_1 = 1, y_j \equiv 1(\text{mod } m),$ where m is the number of different summands, $x_{s+1} = 1, x_s = 2, x_j - x_{j+1} = 2 + \frac{y_{j+2}-1}{m}$	
$\mathcal{F}_1(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^n\,;q)_n}$	$\lambda_1 = 1, \lambda_j - \lambda_{j-1} = 1,$ $y_1 = 1, y_j \equiv 1(\text{mod } m-1),$ where m is the number of different summands, $x_{s+1} = 1, x_s = 2, x_j - x_{j+1} = 2 + \frac{y_{j+2}-1}{m-1}$	
$\mathcal{F}_2(q) = \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q^{n+1}\,;q)_{n+1}}$	$\lambda_1 = 1, \lambda_j - \lambda_{j-1} = 1,$ $y_1 = 1, y_j \equiv 1(\text{mod } m-1),$ where m is the number of different summands, $x_{s+1} = 1 = x_s, x_j - x_{j+1} = 2 + \frac{y_{j+2}-1}{m-1}$	

mock theta function	type $\mathcal{R}$ plane partitions	weight
$S_0(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}(-q\,;q^2)_n}{(-q^2\,;q^2)_n}$	$\lambda_1 = 1, \lambda_{s+1} - \lambda_s \in \{1,2\}$, $y_1 = 1, y_j \equiv 1 (\text{mod } 2)$, $x_{s+1} = 1, x_s = 2, x_j - x_{j+1} = \lambda_{j+2} - \lambda_{j+1} + y_{j+2}$, $\lambda_{j+1} - \lambda_j = 1 + r_j$, where $r_j = \begin{cases} 0, & \text{if } x_j + \lambda_{j+1} - \lambda_j \equiv x_{j+1} + \lambda_{j+2} - \lambda_{j+1} \equiv 1 (\text{mod } 2) \\ 1, & \text{if } x_j + \lambda_{j+1} - \lambda_j \not\equiv x_{j+1} + \lambda_{j+2} - \lambda_{j+1} (\text{mod } 2) \\ 2, & \text{if } x_j + \lambda_{j+1} - \lambda_j \equiv x_{j+1} + \lambda_{j+2} - \lambda_{j+1} \equiv 0 (\text{mod } 2) \end{cases}$	$(-1)^{\frac{1}{2}\sum_{j=2}^{s+1}(y_j - 1)}$
$S_1(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+2)}(-q\,;q^2)_n}{(-q^2\,;q^2)_n}$	$\lambda_1 = 1, \lambda_{s+1} - \lambda_s \in \{1,2\}$, $y_1 = 1, y_{s+1} \geq 3, y_j \equiv 1 (\text{mod } 2)$, $x_{s+1} = 1, x_s = 2, x_j - x_{j+1} = \lambda_{j+2} - \lambda_{j+1} + y_{j+2}$, $\lambda_{j+1} - \lambda_j = 1 + r_j$, where $r_j = \begin{cases} 0, & \text{if } x_j + \lambda_{j+1} - \lambda_j \equiv x_{j+1} + \lambda_{j+2} - \lambda_{j+1} \equiv 1 (\text{mod } 2) \\ 1, & \text{if } x_j + \lambda_{j+1} - \lambda_j \not\equiv x_{j+1} + \lambda_{j+2} - \lambda_{j+1} (\text{mod } 2) \\ 2, & \text{if } x_j + \lambda_{j+1} - \lambda_j \equiv x_{j+1} + \lambda_{j+2} - \lambda_{j+1} \equiv 0 (\text{mod } 2) \end{cases}$	$(-1)^{1+\frac{1}{2}\sum_{j=2}^{s+1}(y_t - 1)}$
$T_0(q) = \sum_{n=0}^{\infty} \frac{q^{(n+1)(n+2)}(-q^2;q^2)_n}{(-q\,;q^2)_{n+1}}$	$\lambda_1 = 1, \lambda_{s+1} - \lambda_s = 1, 3 \leq \lambda_j - \lambda_{j-1} \leq 5$, $y_1 = 1, y_j \geq 1$, $x_{s+1} = 1, x_s = 3$, $x_j - x_{j+1} = \lambda_{j+2} - \lambda_{j+1} + 2y_{j+2} - 3$	$(-1)^{-3s-1+\lambda_{s+1}+\sum_{t=1}^{s} x_t + y_{t+1}}$
$T_1(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}(-q^2\,;q^2)_n}{(-q\,;q^2)_{n+1}}$	$\lambda_1 = 1, 3 \leq \lambda_j - \lambda_{j-1} \leq 5$, $y_1 = 1, y_j \geq 1$, $x_{s+1} = 1, x_s = 1$, $x_j - x_{j+1} = \lambda_{j+2} - \lambda_{j+1} + 2y_{j+2} - 3$	$(-1)^{-3s-1+\lambda_{s+1}+\sum_{t=1}^{s} x_t + y_{t+1}}$
$U_0(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}(-q\,;q^2)_n}{(-q^4\,;q^4)_n}$	$\lambda_1 = 1, \lambda_{s+1} - \lambda_s \in \{1,2\}$, $y_1 = 1, y_j \equiv 1 (\text{mod } 4)$, $x_{s+1} = 1, x_s = 2, x_j - x_{j+1} = \lambda_{j+2} - \lambda_{j+1} + y_{j+2}$, $\lambda_{j+1} - \lambda_j = 1 + r_j$, where $r_j = \begin{cases} 0, & \text{if } x_j + \lambda_{j+1} - \lambda_j \equiv x_{j+1} + \lambda_{j+2} - \lambda_{j+1} \equiv 1 (\text{mod } 2) \\ 1, & \text{if } x_j + \lambda_{j+1} - \lambda_j \not\equiv x_{j+1} + \lambda_{j+2} - \lambda_{j+1} (\text{mod } 2) \\ 2, & \text{if } x_j + \lambda_{j+1} - \lambda_j \equiv x_{j+1} + \lambda_{j+2} - \lambda_{j+1} \equiv 0 (\text{mod } 2) \end{cases}$	$(-1)^{\frac{1}{4}\sum_{j=2}^{s+1}(y_j - 1)}$
$U_1(q) = \sum_{n=0}^{\infty} \frac{q^{(n+1)^2}(-q\,;q^2)_n}{(-q^2\,;q^4)_{n+1}}$	$\lambda_1 = 1, \lambda_{s+1} - \lambda_s = 1$, $y_1 = 1, y_j \equiv 1 (\text{mod } 2)$, $x_{s+1} = 1, x_s = 2, x_j - x_{j+1} = \lambda_{j+2} - \lambda_{j+1} + 2y_{j+2} - 1$, $\lambda_{j+1} - \lambda_j = 1 + r_j$, where $r_j = \begin{cases} 0, & \text{if } x_j + \lambda_{j+1} - \lambda_j \equiv x_{j+1} + \lambda_{j+2} - \lambda_{j+1} \equiv 1 (\text{mod } 2) \\ 1, & \text{if } x_j + \lambda_{j+1} - \lambda_j \not\equiv x_{j+1} + \lambda_{j+2} - \lambda_{j+1} (\text{mod } 2) \\ 2, & \text{if } x_j + \lambda_{j+1} - \lambda_j \equiv x_{j+1} + \lambda_{j+2} - \lambda_{j+1} \equiv 0 (\text{mod } 2) \end{cases}$	$(-1)^{\frac{1}{2}\sum_{j=2}^{s+1}(y_j - 1)}$
$\frac{1}{2}(1 + V_0(q)) = \sum_{n=0}^{\infty} \frac{q^{n^2}(-q\,;q^2)_n}{(q\,;q^2)_n}$	$\lambda_1 = 1, \lambda_{s+1} - \lambda_s\{1,2\}$, $y_1 = 1, y_j \geq 1$, $x_{s+1} = 1, x_s = 1, x_j - x_{j+1} = \lambda_{j+2} - \lambda_{j+1} + 2y_{j+2} - 1$, $\lambda_{j+1} - \lambda_j = 1 + r_j$, where $r_j = \begin{cases} 0, & \text{if } x_j + \lambda_{j+1} - \lambda_j \equiv x_{j+1} + \lambda_{j+2} - \lambda_{j+1} \equiv 1 (\text{mod } 2) \\ 1, & \text{if } x_j + \lambda_{j+1} - \lambda_j \not\equiv x_{j+1} + \lambda_{j+2} - \lambda_{j+1} (\text{mod } 2) \\ 2, & \text{if } x_j + \lambda_{j+1} - \lambda_j \equiv x_{j+1} + \lambda_{j+2} - \lambda_{j+1} \equiv 0 (\text{mod } 2) \end{cases}$	
$\frac{1}{2}(1 + V_0(q)) = \sum_{n=0}^{\infty} \frac{q^{2n^2}(-q^2\,;q^4)_n}{(q\,;q^2)_{2n+1}}$	an even number of summands $\lambda_1 = 1, \lambda_{s+1} - \lambda_s = 1$, $\lambda_{j+1} - \lambda_j \begin{cases} = 1, & \text{if } j \text{ is even} \\ \in \{3,5\}, & \text{if } j \text{ is odd} \end{cases}$ $y_1 = 1, y_j \geq 1$, $x_{s+1} = 1, x_s = 1$, $x_j - x_{j+1} = \begin{cases} 2y_{j+2} - 2, & \text{if } j \text{ is even} \\ \lambda_{j+3} - \lambda_{j+2} + 2y_{j+2} - 3, & \text{if } j \text{ is odd} \end{cases}$	

mock theta function	type $\mathcal{R}$ plane partitions	weight
$V_1(q) = \sum_{n=0}^{\infty} \frac{q^{(n+1)^2}(-q\,;q^2)_n}{(q\,;q^2)_{n+1}}$	$\lambda_1 = 1, \lambda_{s+1} - \lambda_s = 1,$ $y_1 = 1, y_j \geq 1,$ $x_{s+1} = 1, x_s = 1, x_j - x_{j+1} = \lambda_{j+2} - \lambda_{j+1} + 2y_{j+2} - 1,$ $\lambda_{j+1} - \lambda_j = 1 + r_j,$ where $r_j = \begin{cases} 0, & \text{if } x_j + \lambda_{j+1} - \lambda_j \equiv x_{j+1} + \lambda_{j+2} - \lambda_{j+1} \equiv 1 (\text{mod } 2) \\ 1, & \text{if } x_j + \lambda_{j+1} - \lambda_j \not\equiv x_{j+1} + \lambda_{j+2} - \lambda_{j+1} (\text{mod } 2) \\ 2, & \text{if } x_j + \lambda_{j+1} - \lambda_j \equiv x_{j+1} + \lambda_{j+2} - \lambda_{j+1} \equiv 0 (\text{mod } 2) \end{cases}$	
$q^{-1}V_1(q) = \frac{1}{q}\sum_{n=0}^{\infty} \frac{q^{2n^2+2n+1}(-q^4\,;q^4)_n}{(q\,;q^2)_{2n+2}}$	an odd number of summands $\lambda_1 = 1, \lambda_{s+1} - \lambda_s = 1,$ $\lambda_{j+1} - \lambda_j \begin{cases} = 1, & \text{if } j \text{ is odd} \\ \in \{3,5\}, & \text{if } j \text{ is even} \end{cases}$ $y_1 = 1, y_j \geq 1,$ $x_{s+1} = 1, x_s = 1,$ $x_j - x_{j+1} = \begin{cases} 2y_{j+2} - 2, & \text{if } j \text{ is odd} \\ \lambda_{j+3} - \lambda_{j+2} + 2y_{j+2} - 3, & \text{if } j \text{ is even} \end{cases}$	

References

[1] A.K. Agarwal, n-color partition theoretic interpretations of some mock theta functions, Electron. J. Combin. **11** (2004) N14.

[2] A.K. Agarwal, G. Sood, Split $(n+t)$-color partitions and Gordon-McIntosh eight order mock theta functions, Electron. J. Combin. 21 (2014) P2.46.

[3] G.E. Andrews, F.G. Garvan, Dyson's Crank of a Partition, *Bull. Amer. Math. Soc.* **18**(2) (1988) 167–171.

[4] N.D. Baruah, N. M. Begum, *Generating functions and congruences for some partition functions related to mock theta functions*, Int. J. Number Theory **16** (2020), 423–446.

[5] E.H.M. Brietzke, J.P.O. Santos, R. Silva, Bijective proofs using two-line matrix representations for partitions, *Ramanujan J.* **23** (2010) 265–295.

[6] E.H.M. Brietzke, J.P.O. Santos, R. Silva, A new approach and generalizations to some results about mock theta functions, *Discrete Math.* **311** (2011) 595–615.

[7] E.H.M. Brietzke, J.P.O. Santos, R. da Silva, Combinatorial interpretations as two-line array for the mock theta functions, *Bull. Braz. Math. Soc.* **44** (2013) 233–253.

[8] Y-S. Choi, B. Kim, Partition identities from third and sixth order mock theta functions, *European J. Combin*, **33** (2012) 1739–1754.

[9] S.-P. Cui, N.S.S. Gu, *Some new mock theta functions*, Adv. in Appl. Math. **131** (2021), Paper No. 102267.

[10] Godinho, H., Santos, J.P.O., *On a new formula for the number of unrestricted partitions*, Ramanujan J. **55** (2021), 297–307.

[11] J. Lovejoy, R. Osburn, *The Bailey chain and mock theta functions*, Adv. Math. **238** (2013), 442–458.

[12] Gordon, B., McIntosh, R.J., *Some eight order mock theta functions*, J. Lond. Math. Soc. *62* (2000), 321–335.

[13] S. Ramanujan, The Lost Notebook and other Unpublished Papers, Narosa, New Deli, 1988.

[14] J.P.O. Santos, P. Mondek, A.C. Ribeiro, New Two-Lines Arrays Representing Partitions, *Ann. Comb.* **15** (2011) 341–354.

[15] J.K. Sareen, M. Rana, Combinatorics of tenth-order mock theta functions, Proc. Indian Acad. Sci. (Math. Sci.) **126** (2016) 549–556.

[16] S. Sharma, M. Rana, *Combinatorial interpretations of mock theta functions by attaching weights*, Discrete Math. **341** (2018), 1903–1914.

[17] N. J. A. Sloane, *The On-line Encyclopedia of Integer Sequences*, published eletronically at https://oeis.org/.

[18] E.W. Weisstein. Mock Theta Function. From MathWorld?A Wolfram Web Resource. http://mathworld.wolfram.com/MockThetaFunction.html. Accessed 18 May 2016.

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