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ORBITAL STABILITY IN THE TWO-BODY PROBLEM AND KEPLER'S LAWS: A DIDACTIC APPROACH

Doglasse João Mário
Universidade Estadual de Campinas,Brazil.
Ensino Secundário Aberto Moçambicano (ESAM), Massangulo, Província de Niassa, Moçambique.
djoaomario@gmail.com

Handled by A. Carvalho

Abstract
The two-body problem is a fundamental concept in celestial mechanics and theoretical physics, addressing the motion of two particles that interact exclusively through a central force, such as gravity. This article explores the stability of the two-body problem, investigating whether the trajectories of the particles remain confined to a specific region of space over time. The analysis is based on Kepler's laws and the equations of motion derived from Newton's second law. Understanding the stability of this problem is crucial both for the accurate prediction of the behavior of celestial bodies and for practical applications in aerospace engineering and astrophysics.

Keywords: Two-body problem, Orbital stability, Kepler's laws.

1 Introduction

A classical challenge in the field of dynamics is the study of the motion of multiple bodies interacting through gravitational attraction, as governed by Newton's law of universal gravitation. While the general n-body problem remains analytically unsolved, the restricted case of two bodies admits a complete analytical solution. This two-body framework finds widespread application in celestial mechanics, where it successfully models diverse astronomical systems–from planetary orbits around stars to the Moon's motion around Earth, as well as the trajectories of artificial satellites in Earth's orbit. The theoretical foundation provided by the two-body problem has enabled crucial advances in orbital mechanics and space mission design. Recent applications and extensions of this classical problem can be found in the works of [15, 16, 17].The two-body framework successfully models the mentioned astronomical systems to a first approximation. For example, the precessions of the Moon, caused mainly by the non-spherical shape of the Earth, are a somewhat small but visible effects that cannot be studied in this model.

For more complex and realistic scenarios [2, 3], it is necessary to consider orbital perturbations caused by the presence of other celestial bodies. A relevant example is the study of the gravitational effects of massive planets on asteroids in the belt between Mars and Jupiter. Given the impossibility of obtaining exact solutions for such configurations, numerical methods become essential for simulating and predicting dynamic behaviors.

In this context, the two-body problem stands out as a fundamental model, as it allows for the validation of theoretical results and comparison with more elaborate computational approximations.

The two-body problem dates back to Johannes Kepler's (1609) [7] observations regarding the orbit of Mars, where he found discrepancies of 8 arcminutes between Tycho Brahe's data and the Copernican circular model. The first scientifically accepted theoretical formulation was imposed by Isaac Newton (1687) [11, 10] through his law of universal gravitation.

Kepler's laws, derivable from Newtonian equations, provide the theoretical framework for stable orbits (elliptical, parabolic, or hyperbolic). However, real systems experience perturbations.

Therefore, our objective in this review article is to present a basic and didactic text on the two-body problem. Other texts with a similar approach can be found in [20, 9].

Thus, the text is organized into sections. In the first, we address central forces, defining gravitational force, the Lagrangian, momentum, and the potential of the two-body system. Next, we study orbital stability based on the equations obtained in the previous section, including Binet's differential equation, which expresses the relationship between the radius r and the angle ϕ of an orbit around a force center, allowing for the description of conic sections. Additionally, we dedicate a section to discussing Kepler's laws and their evidence in the two-body problem. Finally, we present the conclusions of this study.

Although the two-body problem is fundamental for understanding phenomena such as dumbbell dynamics [5, 6, 1] and tides [4, 14], these topics will not be addressed in this work, being reserved for future studies.

2 Central Forces

In this section, we investigate the basic foundations of the two-body problem interacting through central forces, which serves as the fundamental framework for understanding diverse phenomena from planetary motion to binary star systems.

A force field in the two-body problem, $\vec{F}$, is said to be central, with the center at $\vec{0}$, if it has the form $\vec{F}(\vec{r}) = F(r)\hat{r}$, where we assume that $F : \mathbb{R} \longrightarrow \mathbb{R}$ is a smooth function. The vector $\vec{r}$ denotes the relative position between the bodies, and $r = |\vec{r}|$ is its magnitude.

These forces are characterized by their radial nature, meaning they act along the direction that connects the positions of the two bodies at each instant [8]. A fundamental aspect of central forces is that their intensity depends solely on the distance between the particles, unaffected by the direction or relative orientation [18]. Consider a system consisting of two isolated bodies as illustrated in Figure 1. The intensity of the interaction force that body two, with mass m_2, experiences due to body one, with mass m_1, is expressed by the law of universal gravitation [10]

$$F = G_U \frac{m_1 m_2}{|\vec{r}_2 - \vec{r}_1|^2},$$

where $\vec{r}_j$, $j = 1, 2$, are the positions of the bodies and $G_U = 6.6743 \times 10^{-11}\,\mathrm{m}^3\,\mathrm{kg}^{-1}\,\mathrm{s}^{-2}$ is the universal gravitational constant.

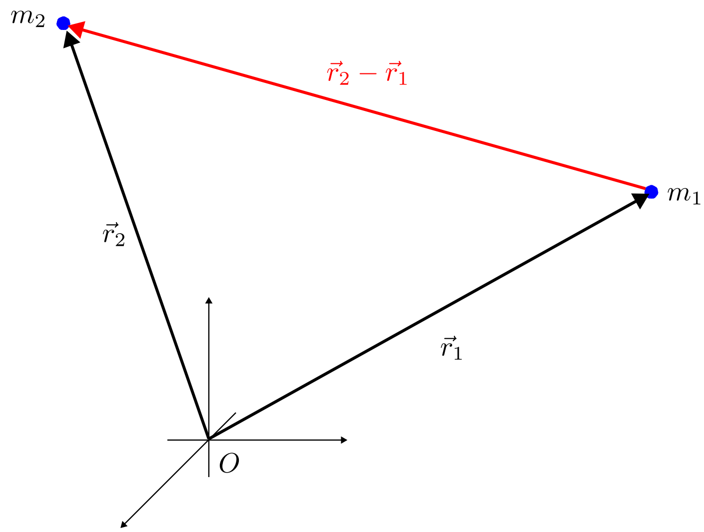

Figure 1: Representation of the system with two masses, m_1 and m_2, and the law of universal gravitation.

As we know, the force is a vector quantity and acts in the direction that connects the two bodies, we can write it in the form

$$\vec{F} = -G_U \frac{m_1 m_2}{|\vec{r}_2 - \vec{r}_1|^3}(\vec{r}_2 - \vec{r}_1). \tag{2.1}$$

This law of forces is conservative, meaning it is given by the gradient of a scalar function, the potential energy of the system, $\vec{F}(\vec{r}) = -\nabla U(r)$, which has the form

$$U(r) = -G_U \frac{m_1 m_2}{r},$$

where the expression (2.1) is obtained with the identification $\vec{r} = \vec{r}_2 - \vec{r}_1$, $r = |\vec{r}_2 - \vec{r}_1|$.

2.1 The Lagrangian of the System

Throughout this work, we will rely on the Lagrangian formalism to derive the equations of motion for the problem at hand. In doing so, we will illustrate how the simplest equations governing the two-body problem can be systematically obtained from this approach.

To facilitate the approach, we will use the notation $\vec{r}$ as the relative position vector, and $\vec{R} = (m_1\vec{r}_1 + m_2\vec{r}_2)/(m_1 + m_2)$ will denote the position of the center of mass of the two-body system.

Proposition 2.1. *The Lagrangian of the two-body problem, represented in Figure 1, is given by*

$$\mathcal{L} = \frac{1}{2} M \dot{\vec{R}}^2 + \frac{1}{2}\mu \dot{\vec{r}}^2 - U(r), \tag{2.2}$$

the total mass M is given by (2.3)

$$M = m_1 + m_2, \tag{2.3}$$

and μ is the reduced mass of the system given by (2.4)

$$\mu = \frac{m_1 m_2}{M}. \tag{2.4}$$

Proof. We know that the Lagrangian is defined as the difference between the kinetic energy and the potential energy, and its expression is written as

$$\mathcal{L} = \frac{1}{2} m_1 \dot{\vec{r}}_1{}^2 + \frac{1}{2} m_2 \dot{\vec{r}}_2{}^2 - U(r), \tag{2.5}$$

The position vector of the center of mass is given by

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}, \tag{2.6}$$

Now, we can express the vectors $\vec{r}_1$ and $\vec{r}_2$ in terms of r and $\vec{R}$ as

$$\begin{cases} \vec{r}_1 = \vec{R} - \frac{m_2}{M}\vec{r}, \\ \vec{r}_2 = \vec{R} + \frac{m_1}{M}\vec{r}, \end{cases} \tag{2.7}$$

Substituting the equations defined in (2.7) into the expression (2.5), we get

$$\mathcal{L} = \frac{1}{2}m_1\left(\dot{\vec{R}} - \frac{m_2}{M}\dot{\vec{r}}\right)^2 + \frac{1}{2}m_2\left(\dot{\vec{R}} + \frac{m_1}{M}\dot{\vec{r}}\right)^2 - U(r),$$

$$\mathcal{L} = \frac{1}{2}m_1\left[\dot{\vec{R}}^2 + \left(\frac{m_2}{M}\right)^2\dot{\vec{r}}^2 - 2\frac{m_2}{M}\dot{\vec{R}}\dot{\vec{r}}\right] + \frac{1}{2}m_2\left[\dot{\vec{R}}^2 + \left(\frac{m_1}{M}\right)^2\dot{\vec{r}}^2 + 2\frac{m_1}{M}\dot{\vec{R}}\dot{\vec{r}}\right] - U(r),$$

$$\mathcal{L} = \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}\frac{m_1m_2(m_1+m_2)}{M^2}\dot{\vec{r}}^2 - U(r).$$

The Lagrangian defined in (2.5) can be rewritten more simply as

$$\mathcal{L} = \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}\frac{m_1m_2}{M}\dot{\vec{r}}^2 - U(r). \tag{2.8}$$

By using the definition of the reduced mass, we see that (2.8) is equivalent to (2.2). □

We see that the Lagrangian (2.8) does not depend on $\vec{R}$, but rather on $\dot{\vec{R}}$, and therefore it is an ignorable or cyclic coordinate. In such cases, there is a conservation law associated with this coordinate. From (2.8), we can see that the Lagrangian of the two-body problem is the sum of the Lagrangian of the center of mass ($\mathcal{L}_{CM}$) and the Lagrangian defined by the relative coordinate ($\mathcal{L}_{Rel}$), where

$$\begin{cases} \mathcal{L}_{CM}(\dot{\vec{R}}) = \frac{1}{2}M\dot{\vec{R}}^2, \\ \mathcal{L}_{Rel}(\dot{\vec{r}}, \vec{r}) = \frac{1}{2}\mu\dot{\vec{r}}^2 - U(r). \end{cases} \tag{2.9}$$

Thus, we can conclude that the equations of motion for these coordinates are decoupled, allowing us to focus on the differential equations for the relative position vector, as the equation of the center of mass only imposes that it moves in uniform rectilinear motion.

2.2 Angular Momentum

Angular momentum is defined as the cross product of the relative position with the linear momentum of a body with respect to a reference point. In the two-body problem, the total angular momentum of the system is, by definition, the sum of the individual angular momenta of each body [**?**].

For central force fields, we have the following law of conservation of angular momentum.

Proposition 2.2. *The angular momentum, with respect to the center of the force, of a particle subject to only a central force is constant throughout the motion.*

Proof. For a system described by a set of generalized coordinates q_i, with $i = 1, \cdots, n$, and a Lagrangian function $\mathcal{L}(q_1, \cdots, q_n, \dot{q}_1, \cdots, \dot{q}_n)$, we know that the equations of motion are derived from the Euler-Lagrange equations [12]

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) = \frac{\partial \mathcal{L}}{\partial q_i}, \quad i = 1, \cdots, n. \tag{2.10}$$

Thus, applying (2.10) to (2.8), and considering that q_i are $\vec{R}$ and $\vec{r}$, we obtain the differential equations

$$\begin{cases} M\ddot{\vec{R}} = 0. \\ \mu\ddot{\vec{r}} = \vec{F}(\vec{r}) = F(r)\hat{r}. \end{cases} \tag{2.11}$$

On the other hand, the angular momentum vector is given by

$$\vec{G} = \vec{r} \times \vec{p}, \tag{2.12}$$

where $\vec{r}$ is the position vector from the center of forces to the particle and (2.13) is its linear momentum. Thus

$$\vec{p} = \mu\dot{\vec{r}} \tag{2.13}$$

and

$$\frac{d\vec{G}}{dt} = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt} = \vec{r} \times \mu\ddot{\vec{r}} = \vec{r} \times \vec{F} = r\hat{r} \times F(r)\hat{r} = \vec{0}.$$

This shows that $\vec{G}$ is constant, which means that the motion of the particle under study always occurs in a fixed plane orthogonal to the constant vector $\vec{G}$, containing the center of forces. In terms of the magnitude of the angular momentum, the equation (2.12) can be written in polar coordinates with the following expression for the velocity vector: $\dot{\vec{r}} = \dot{r}\hat{r} + r\dot{\phi}\hat{\phi}$. Therefore, we have $\vec{G} = \vec{r} \times \mu\vec{v} = \mu\vec{r} \times \dot{\vec{r}} = \mu r\hat{r} \times (\dot{r}\hat{r} + r\dot{\phi}\hat{\phi}) = \mu r^2\dot{\phi}\hat{r} \times \hat{\phi} = \mu r^2\dot{\phi}\hat{k}$, which is constant. We conclude that the following scalar, the magnitude of the angular momentum $G = \|\vec{G}\|$, is also constant

$$G = \mu r^2\dot{\phi}. \tag{2.14}$$

□

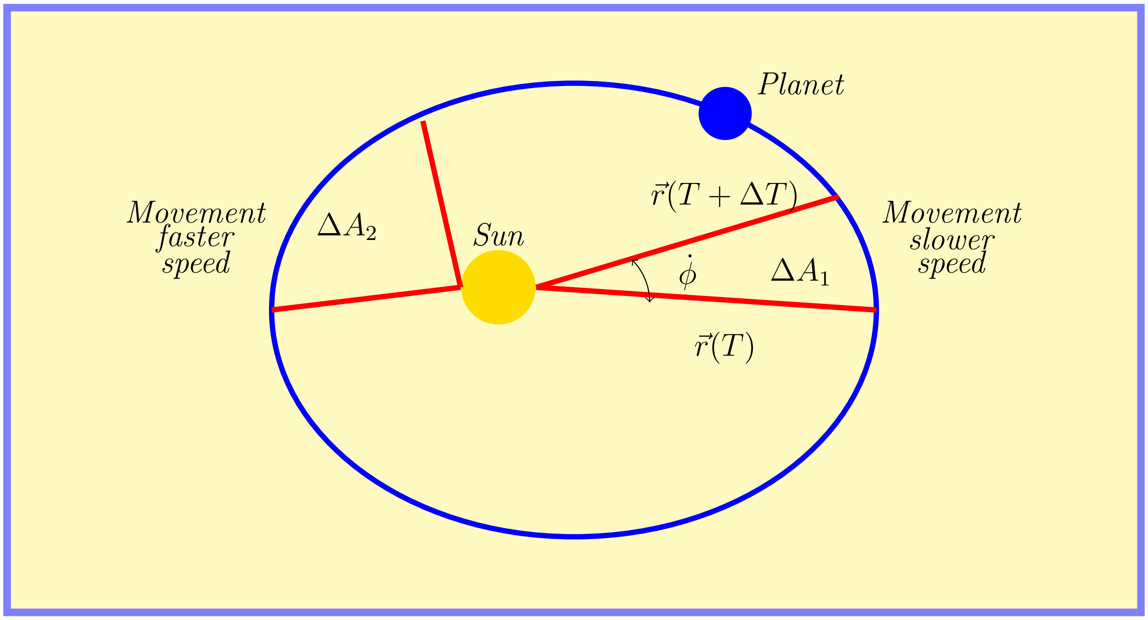

Figure 2: Representation of Kepler's second law where a planet is moving in an elliptical orbit around the Sun.

Central force fields admit orbits with special characteristics, in particular, all of them satisfy the well-known second law of Kepler (Proposition (2.3)), which is a consequence solely of the conservation of angular momentum.

Proposition 2.3. *The segment between the particle and the center of the force sweeps out equal areas in equal times.*

Proof. Consider a body under the action of a central force field, with position $\vec{r}(T)$ at time T, and position $\vec{r}(T + \Delta T)$ at time $T + \Delta T$, as illustrated in Figure 2. The sector swept out by the body's trajectory in this time period, with area ΔA, can be parameterized by

$$\vec{\xi}(u, t) := u\hat{r}(t),$$

where $\hat{r}(t) = \vec{r}/r$ is the unit vector along the radial direction, $0 < u < r(t)$ is the radial distance, and the polar angle $\phi(t)$ varies from $\beta_1 = \phi(T)$ to $\beta_2 = \phi(T + \Delta T)$.

To calculate the area, we use the tangent vectors:

$$\frac{\partial \vec{\xi}}{\partial u} = \hat{r}(t), \quad \frac{\partial \vec{\xi}}{\partial t} = u\dot{\phi}(t)\hat{\phi}(t),$$

where $\hat{\phi}(t)$ is the unit vector orthogonal to $\hat{r}(t)$ in the direction of increasing ϕ. The area ΔA is given by

$$\begin{aligned}\Delta A = \int_T^{T+\Delta T}\int_0^{r(t)}\left\|\frac{\partial\vec{\xi}}{\partial u}\times\frac{\partial\vec{\xi}}{\partial t}\right\| dudt = \int_T^{T+\Delta T}\int_0^{r(t)}\left\|u\hat{r}(t)\times\dot{\phi}(t)\hat{\phi}(t)\right\| dudt \\ = \int_T^{T+\Delta T}\int_0^{r(t)}\left\|u\dot{\phi}(t)\hat{r}(t)\times\hat{\phi}(t)\right\| dudt = \int_T^{T+\Delta T}\int_0^{r(t)} u\dot{\phi}(t)dudt. \quad (2.15)\end{aligned}$$

In (2.15), using the property $\hat{r}(t)\times\hat{\phi}(t) = \hat{z}$, where $\hat{z}$ is a unit vector perpendicular to the plane, the magnitude simplifies

$$\left\|\hat{r}(t)\times\hat{\phi}(t)\right\| = \|\hat{z}\| = 1.$$

After that, we integrate with respect to u and we can write

$$\Delta A = \frac{1}{2}\int_T^{T+\Delta T} r(t)^2\dot{\phi}(t)dt = \frac{1}{2}\int_T^{T+\Delta T}\frac{G}{\mu}dt = \frac{G}{\mu}\Delta T. \quad (2.16)$$

We conclude that the area variation is a constant multiple of the time variation, demonstrating the result. □

The relation (2.16) implies that a planet's velocity is not constant along its orbit [13]. It moves faster when it is closer to the Sun and slower when it is farther away. This law is a consequence of the conservation of angular momentum and plays a fundamental role in understanding the motion of planets around the Sun.

2.3 Potential Energy and Work Done

A force field $\vec{F}(\vec{r})$ is called conservative if there exists a differentiable function $U : \mathbb{R}^3 \to \mathbb{R}$ such that $\vec{F} = -\nabla U$. Orbits generated by conservative fields can be understood through the conservation of mechanical energy, as described below.

Let us calculate the work done by a particle moving from a point A to a point B along a path C. The work done by the particle along this path, in spherical coordinates, is given by:

$$W^C_{A\to B} = \int_{A,C}^{B}\vec{F}\cdot d\vec{r}.$$

In spherical coordinates, the differential displacement $d\vec{r}$ can be written as

$$d\vec{r} = dr\hat{r} + r\sin\theta d\phi\hat{\phi} + rd\theta\hat{\theta},$$

where $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi]$ are the polar and azimuthal angles, respectively. Since the force $\vec{F} = F(r)\hat{r}$ acts radially, the dot product simplifies to

$$\vec{F} \cdot d\vec{r} = F(r)(\hat{r} \cdot \hat{r})dr + F(r)(\hat{r} \cdot \hat{\phi})(r \sin\theta d\phi) + F(r)(\hat{r} \cdot \hat{\theta})(rd\theta).$$

Here, the orthogonality of the unit vectors $\hat{r}$, $\hat{\phi}$, and $\hat{\theta}$ implies

$$\hat{r} \cdot \hat{\phi} = 0, \quad \hat{r} \cdot \hat{\theta} = 0, \quad \hat{r} \cdot \hat{r} = 1$$

leaving only the term $F(r)(\hat{r} \cdot \hat{r})dr = F(r)dr$. Thus, the work done is

$$W^C_{A\to B} = \int_{A,C}^{B} F(r)dr.$$

This shows that the work depends only on the radial displacement. We can now relate it to the change in potential energy

$$W^C_{A\to B} = -\Delta U = -(U(\vec{r}_B) - U(\vec{r}_A)).$$

Therefore, the force $\vec{F}$ is conservative, and the potential energy is given by

$$U(\vec{r}) = -\int_{r_A}^{r} F(r)dr, \quad r = |\vec{r}|. \tag{2.17}$$

The total mechanical energy of the two-body problem under the action of a central force field is defined by

$$E = \frac{1}{2}\mu\dot{\vec{r}}^2 + U(r), \tag{2.18}$$

where U is the potential energy defined in (2.17).

Proposition 2.4. *The mechanical energy of the two-body problem can be equivalently calculated as*

$$E = \frac{1}{2}\mu\dot{r}^{\,2} + V_{eff}(r), \tag{2.19}$$

which is conserved throughout the motion, where V_{eff} is the effective potential, defined in (2.20)*.*

Proof. The mechanical energy of the system is given by (2.18). Its variation along the solutions is given by

$$\frac{dE}{dt} = \mu\dot{\vec{r}} \cdot \ddot{\vec{r}} + \nabla U(\vec{r}) \cdot \dot{\vec{r}} = \left(\mu\ddot{\vec{r}} + \nabla U(\vec{r})\right) \cdot \dot{\vec{r}} = 0,$$

where the last equality follows from the equations of motion (2.11). Thus, the function E is constant along the solutions.

Using equations (2.14) and (2.17), we can state that, for planar motion,

$$E = \frac{1}{2}\left[\left(\dot{r}\hat{r} + r\dot{\phi}\hat{\phi}\right)\left(\dot{r}\hat{r} + r\dot{\phi}\hat{\phi}\right)\right] + U(r) = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\mu r^2\dot{\phi}^2 + U(r) = \frac{1}{2}\mu\dot{r}^2 + \frac{G^2}{2\mu r^2} + U(r),$$

which represents the mechanical energy for the two-body problem. Therefore, if we consider

$$V_{\text{eff}} = \frac{G^2}{2\mu r^2} + U(r), \tag{2.20}$$

we conclude that $E = K + V_{\text{eff}}$. Thus, we obtain

$$E = \frac{1}{2}\mu\dot{r}^2 + V_{\text{eff}}(r). \tag{2.21}$$

□

Thus, we can see that the central force problem reduces to two one-dimensional problems, for each coordinate, which can be studied using the conservation of energy and angular momentum equations

$$\begin{cases} \frac{1}{2}\mu\dot{r}^2 + V_{\text{eff}}(r) = E, \\ \mu r^2\dot{\phi} = G. \end{cases} \tag{2.22}$$

3 Orbital Stability

This section deals with theoretical stability criteria, showing why planetary orbits (such as the Earth's orbit around the Sun) resist small perturbations. The structure extends to exotic potentials, with implications for celestial mechanics.

We recall that our main problem is to describe the dynamics of the particle with reduced mass μ, with an effective potential energy given by

$$V_{\text{eff}} = \frac{G^2}{2\mu r^2} + U(r), \tag{3.1}$$

where r is the distance between the particle and the center of the force, G is the magnitude of the angular momentum of the particle relative to the center of the force, and $U(r)$ is the interaction potential energy of the particle relative to the center of the force. Moreover, the effective force is given by

$$F_{\text{eff}}(r) = -\frac{d}{dr}V_{\text{eff}}(r) = \frac{G^2}{\mu r^3} + F(r). \tag{3.2}$$

We define the orbit of the two-body problem as any solution of the differential equation (2.22). We note that these orbits can also be determined by the relationship between the two polar coordinates, independent of time, in the form $r = r(\phi)$. There exist special orbits, called circular orbits, in which the distance from the body to the center of the force is constant, that is, $r(\phi) = r_0$. These orbits are given by the solutions of (2.22) with initial conditions $\dot{r}(0) = 0$ and $r(0) = r_0$, where the latter is a root of the equation $F_{\text{eff}}(r_0) = 0$.

3.1 Stability Condition

For each circular orbit in a two-body problem, we can inquire about its stability. The general idea is to declare an orbit stable if small perturbations do not significantly alter the behavior of its nearby perturbed orbits. On the other hand, a circular orbit should be considered unstable if small perturbations can cause significant deviations from the original orbit. In this section, we study the stability of general central forces, but our main example is the Newtonian force, i.e., with potential $U(r) = -k/r$, where $k = G_U m_1 m_2 > 0$.

Under these assumptions, from equation (3.1) we know that

$$V_{\text{eff}}(r) = \frac{G^2}{2\mu r^2} - \frac{k}{r}. \tag{3.3}$$

For the function (3.3), we consider the parameters $k = \mu = L = 1$, and we display its graph in Figure 3. Since the effective force satisfies $F_{\text{eff}}(r) = -V'_{\text{eff}}(r)$, we are led to consider a circular orbit stable if its radius is a local minimum of $V'_{\text{eff}}(r)$; otherwise, if it is a local maximum or inflection point, we have an unstable circular orbit. Thus, a sufficient condition for a circular orbit with radius $r_0 > 0$ to be stable is that $-F_{\text{eff}}(r_0) = V'_{\text{eff}}(r_0) = 0$ and $V''_{\text{eff}}(r_0) > 0$.

For an arbitrary central potential $U(r)$, a stable circular orbit satisfies the conditions $V'_{\text{eff}}(r_0) = -\dfrac{G^2}{\mu r_0^3} + U'(r_0) = 0$ and $V''_{\text{eff}}(r_0) = \dfrac{3G^2}{\mu r_0^4} + U''(r_0) > 0$. Therefore, the stability condition can be rewritten as

$$\frac{3}{r_0}U'(r_0) + U''(r_0) > 0. \tag{3.4}$$

The stability of a circular orbit also coincides with the condition of local energy minimum for fixed angular momentum levels, as proved in Proposition (3.1).

Proposition 3.1. *Suppose $r_0 > 0$ is a local minimum of $V_{\mathit{eff}} = \frac{G^2}{2\mu r^2} + U(r)$, then $(r_0, 0) \in \mathbb{R}^2$ is a local minimum of $E(r, \dot{r}) = \frac{1}{2}\mu\dot{r}^2 + V_{\mathit{eff}}(r) = \frac{1}{2}\mu\dot{r}^2 + \frac{G^2}{2\mu r^2} + U(r)$.*

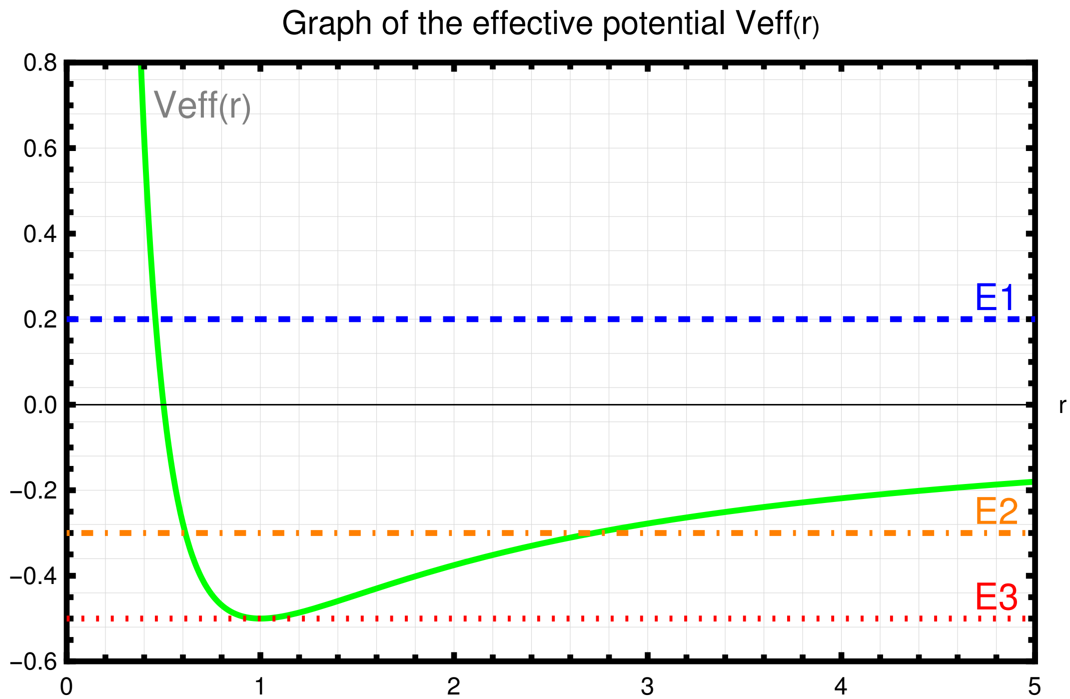

Figure 3: Effective potential for the Newtonian potential $U = -k/r$. The only present circular orbit is stable.

Proof. If r_0 is a local minimum of $V_{\text{eff}}(r)$, $V'_{\text{eff}}(r_0) = 0$ and $V''_{\text{eff}}(r_0) > 0$. Moreover, the gradient of the energy function is

$$\nabla E(r, \dot{r}) = (V'_{\text{eff}}(r), \mu\dot{r}) \Rightarrow \nabla E(r_0, 0) = (V'_{\text{eff}}(r_0), 0) = (0, 0),$$

whose Hessian matrix at this critical point is

$$\text{Hess}E(r_0, 0) = \begin{bmatrix} V''_{\text{eff}}(r_0) & 0 \\ 0 & \mu \end{bmatrix}.$$

Since the eigenvalues of $\text{Hess}E(r_0, 0)$ are $V''_{\text{eff}}(r_0) > 0$ and $\mu > 0$, it follows that the critical point $(r_0, 0)$ is a local minimum of $E(r, \dot{r})$. $\square$

Now, consider an attractive potential in the form of a power-law, $U(r) = -kr^n$, where $n \neq 0$ and $k \neq 0$ so that $F(r) = -U'(r) = nkr^{n-1} < 0$ for all $r > 0$. We wish to determine for which values of n the function $U(r)$ generates stable circular orbits. To do so, we apply the stability condition given by equation (3.4). Observe that

$U''(r_0) = -n(n-1)kr_0^{n-2}$, and therefore

$$\frac{3}{r_0}U'(r_0) + U''(r_0) = -3nkr_0^{n-2} - n(n-1)kr_0^{n-2} = -nkr_0^{n-2}(2+n). \tag{3.5}$$

Since $nk < 0$, we see that the stability condition (3.4) is equivalent to $n > -2$. In particular, for the Newtonian potential $U = -kr^{-1}$, all of its circular orbits are stable. The radii of the circular orbits are the roots of $V'_{\text{eff}}(r_0) = -\dfrac{G^2}{\mu r_0^3} - nkr_0^{n-1} = 0$, that is,

$$r_0 = \left(\frac{G^2}{\mu(-nk)}\right)^{\frac{1}{n+2}}. \tag{3.6}$$

Thus, there is exactly one circular orbit radius for each $n \neq 0$ and $n \neq -2$. In particular, for the Newtonian potential, $r_0 = \mu k/G^2$, represented in Figure 3.

4 Binet's Equation

This section integrates the general theory of central forces discussed in Section 2 with specific examples, including non-Newtonian potentials. The primary outcome is the Binet Equation.

Using a more direct approach, based on Newton's second law and the second equation of (2.11), we can also obtain a direct differential equation for the orbit. It is a consequence of energy conservation (2.19), where $\dot{E} = 0$ is equivalent to

$$\mu\ddot{r} = \frac{G^2}{\mu^2 r^3} + F(r). \tag{4.1}$$

Let us make a change of variables, where

$$u = \frac{1}{r}, \tag{4.2}$$

then we have

$$\frac{dr}{dt} = \frac{dr}{du}\frac{du}{d\phi}\frac{d\phi}{dt} = -\frac{1}{u^2}\frac{du}{d\phi}\frac{Gu^2}{\mu} = -\frac{G}{\mu}\frac{du}{d\phi}. \tag{4.3}$$

Thus,

$$\ddot{r} = \frac{d}{dt}\left(-\frac{G}{\mu}\frac{du}{d\phi}\right) = -\frac{G}{\mu}\frac{d}{d\phi}\left(\frac{du}{d\phi}\right)\frac{d\phi}{dt} = -\frac{G^2}{\mu^2}u^2\frac{d^2u}{d\phi^2}.$$

Therefore, equation (4.1) can be rewritten as follows:

$$-\frac{G^2}{\mu}u^2\frac{d^2u}{d\phi^2} - \frac{G^2}{\mu^2 r^3} = F(r), \tag{4.4}$$

and by using equation (4.2), we get

$$-\frac{G^2}{\mu}u^2\frac{d^2u}{d\phi^2} - \frac{G^2}{\mu}u^3 = F\left(\frac{1}{u}\right),$$

therefore

$$\frac{d^2u}{d\phi^2} + u = -\frac{\mu}{G^2u^2}F\left(\frac{1}{u}\right). \tag{4.5}$$

Equation (4.5) is known as the differential equation of the orbit or Binet's equation [19].

Indeed, considering a particle of mass μ under the influence of a central force with an orbit $r = ke^{\alpha\phi}$, where $k \neq 0$ and $\alpha \neq 0$, we can determine the force responsible for generating this orbit using Binet's equation. We write $u = \frac{1}{r} = \frac{e^{-\alpha\phi}}{k}$, and thus, we have $\dot{u} = -\alpha u$, $\ddot{u} = \alpha^2 u$. Substituting these expressions into Binet's equation (4.5), we get

$$\alpha^2 u + u = -\frac{\mu}{G^2u^2}F \Rightarrow F = -\frac{(\alpha^2+1)G^2}{\mu}u^3 \Rightarrow F(r) = -\frac{G^2}{\mu}(\alpha^2+1)\frac{1}{r^3}.$$

As we can see, this represents an attractive force that decays with distance more rapidly than the gravitational force.

4.1 Conic Sections

To understand the geometry of the orbits in the Kepler problem, we briefly review the equation of conics in polar coordinates.

A conic is constructed from a point F, its focus, and a line ℓ, the directrix, where $F \notin \ell$. In Figure 4, we see a polar representation where ℓ represents the directrix and F is the focus. Additionally, this representation is made assuming the focus is located at the origin of the plane. The conic is defined as the set of points Q that satisfy the following equation

$$\frac{d(F,Q)}{d(Q,\ell)} = e > 0, \tag{4.6}$$

where $d(\cdot,\cdot)$ denotes the Euclidean distance, and e is a fixed constant, the eccentricity of the conic. For $0 < e < 1$, the conic is bounded, an ellipse. For $e = 1$, the conic is a parabola, and for $e > 1$, the conic is a hyperbola.

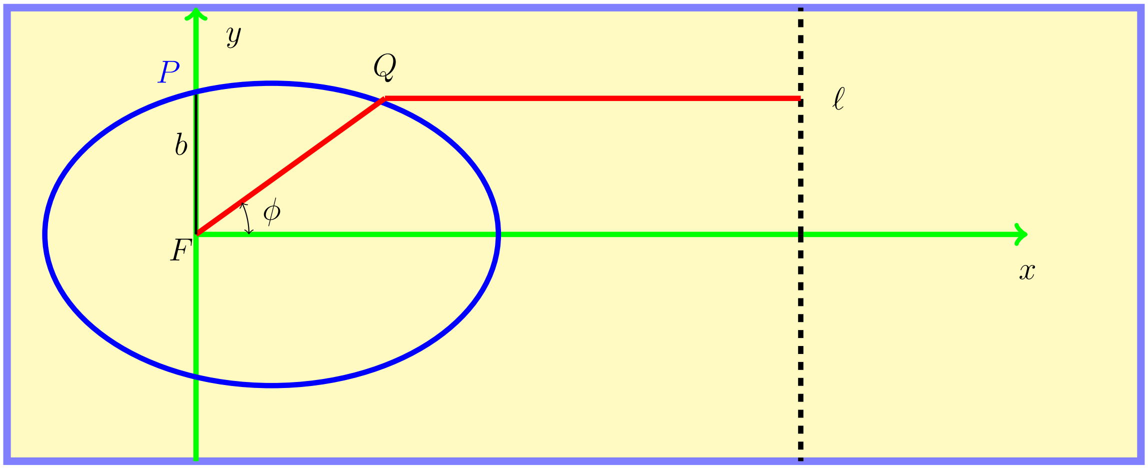

Figure 4: Polar representation of conic sections.

Looking at Figure 4, and denoting by P the intersection of the conic with the y-axis, above the origin, with $b = d(F, P)$, we note that

$$\frac{b}{d(P,\ell)} = e \Rightarrow b = e(r\cos\phi + d(Q,\ell)) = er\cos\phi + ed(Q,F) = er\cos\phi + r,$$

where $Q = (r\cos\phi, r\sin\phi)$ is an arbitrary point on the conic. Hence, we obtain the equation

$$r = \frac{b}{1 + e\cos\phi}, \quad \forall\,\phi \geq 0, \tag{4.7}$$

which is the polar representation of conics.

Note that in the case $e = 0$, the equation reduces to a constant $r = b$, corresponding to a circle. However, the definition given in (4.6) becomes ill-defined because the directrix ℓ no longer plays a role in the geometry of a circle. This special case highlights the limitation of using the directrix-focus definition for circular conics.

5 Kepler Problem

This section integrates the mathematical framework established earlier to rigorously demonstrate Kepler's three laws. Key insights include the derivation of the general orbit from Binet's equation, the relationship between energy and orbital characteristics, and the resulting physical predictions.

Although the Newtonian potential $U(r) = -\frac{k}{r}$ was introduced and analyzed in Subsection 3.1, we now focus specifically on its connection to Kepler's laws and the resulting orbital dynamics. In this case, Kepler's three laws hold.

Kepler's First Law: The bounded orbits of planets are elliptical, with the Sun occupying one of the foci of the ellipse.

Kepler's Second Law: The radial vector from the Sun to the planet sweeps out equal areas in equal intervals of time.

Kepler's Third Law : The square of the orbital period is directly proportional to the cube of the semi-major axis of the ellipse.

Kepler's Second Law was previously proven for an arbitrary attractive central force. We will now prove the remaining laws as immediate consequences of the solutions to Binet's equation.

For the law of universal gravitation, it was proven earlier that there is only one equilibrium radius, given by

$$r_0 = \frac{G^2}{\mu k}. \tag{5.1}$$

We also verified that this solution is stable, with minimum energy for each positive fixed value of angular momentum. The minimum energy associated with this circular orbit is given by

$$E_0 = U_{\text{eff}}(r_0) = -\frac{\mu k^2}{2G^2} < 0. \tag{5.2}$$

We will also find the return points for a given energy $E > E_0$, where the derivative of the relative distance vanishes, i.e.,

$$U_{\text{eff}}(r_R) = \frac{G^2}{2\mu r_R^2} - \frac{k}{r_R} = E,$$

therefore

$$r_R = \frac{r_0}{1 \pm \sqrt{1 - \frac{E}{E_0}}}. \tag{5.3}$$

With this set of information, we can determine the types of trajectories. For this, we use Binet's equation with the force law $F = -k/r^2 = -ku^2$. Substituting into equation (4.5), we obtain

$$\ddot{u} + u = \frac{\mu}{G^2}\frac{F}{u^2} = \frac{\mu}{G^2}k = \frac{1}{r_0}. \tag{5.4}$$

Equation (5.4) is a second-order, linear, non-homogeneous ordinary differential equation. Thus, its solution is the sum of the homogeneous solution and the particular solution, namely

$$u_H(\phi) = A\cos(\phi+\gamma), \qquad u_P = \frac{1}{r_0}. \tag{5.5}$$

Thus, the general solution to (5.4) is the sum of $u_H(\phi)$ and $u_P(\phi)$,

$$u(\phi) = A\cos(\phi+\gamma) + \frac{1}{r_0} \Rightarrow \frac{1}{r(\phi)} = A\cos(\phi+\gamma) + \frac{1}{r_0},$$

which gives

$$r = \frac{1}{A\cos(\phi+\gamma) + \dfrac{1}{r_0}} \Rightarrow r = \frac{r_0}{1 + r_0 A\cos(\phi+\gamma)}.$$

Comparing the last expression with equation (4.7), we see that the orbit is a conic. The angle γ only locates the orientation of its axes, and it can be taken as $\gamma = 0$. The general solution is then

$$r = \frac{r_0}{1 + r_0 A\cos\phi}. \tag{5.6}$$

We can observe that the eccentricity in (5.6) is given by $e = |r_0 A|$, and using equation (5.3), we can write $r_R = \frac{r_0}{1 \pm r_0 A}$. Therefore, the eccentricity is given by

$$e = \sqrt{1 - \frac{E}{E_0}}. \tag{5.7}$$

In this way, we can classify all possible solutions, with non-zero angular momentum.

- If $E = E_0$, then $e = 0$. The conic section is a circle.
- If $E_0 < E < 0$, then $0 < e < 1$. The conic section is an ellipse, as shown in Figure 3 for energies E_2 and E_3.
- If $E = 0$, then $e = 1$. There is a single return point, and the conic section is a parabola.
- If $E > 0$, then $e > 1$. In this case, the orbit is a hyperbola, as depicted in Figure 3 for energy E_1.

Kepler's First Law, is an immediate consequence of the conclusions above. By Kepler's Second Law, equation (2.15), we know that the angular momentum G and the orbital period are related by

$$G = \frac{2\mu A}{\Delta T} = \frac{2\mu\pi a^2\sqrt{1-e^2}}{\Delta T}, \tag{5.8}$$

where ΔT denotes the full period of the elliptical orbit, and we have used the formula for the area of the ellipse, $A = \pi ab$, with $a \geq a\sqrt{1-e^2} = b$ being the semi-major and semi-minor axes. From the orbit equation (5.6), when the body is at the aphelion $\phi = 0$, we have

$$a(1-e) = \frac{r_0}{1+e} = \frac{G^2}{\mu k(1+e)} \Rightarrow G^2 = a(1-e^2)\mu k. \tag{5.9}$$

From equations (5.8) and (5.9), we obtain Kepler's Third Law:

$$(\Delta T)^2 = \frac{4\pi^2\mu}{k}a^3. \tag{5.10}$$

These are the classic results of the Kepler problem. In the following table we can see an example using the theoretical constants of the solar system.

Celestial Body	a (AU)	T (years)	$(\Delta T)^2$ (years^2)	a^3 (AU^3)	$(\Delta T)^2/a^3$
Mercury	0.387	0.241	0.058	0.058	1.000
Venus	0.723	0.615	0.378	0.378	1.000
Earth	1.000	1.000	1.000	1.000	1.000
Mars	1.524	1.881	3.540	3.540	1.000
Jupiter	5.203	11.862	140.9	140.9	1.000
Saturn	9.537	29.457	867.0	867.0	1.000
Uranus	19.191	84.020	7059.4	7059.4	1.000
Neptune	30.068	164.8	27183	27183	1.000

Table 1: Data illustrating Kepler's Third Law for the planets of the Solar System, normalized such that the ratio $(\Delta T)^2/a^3$ is constant.

6 Conclusions

The stability analysis was conducted exclusively for circular orbits, which represent an equilibrium state of the system.

Orbital stability in the two-body problem is a central theme in celestial mechanics and has been largely influenced by Kepler's Laws. This problem addresses the gravitational interaction between two point masses, assuming the absence of other acting forces. The solutions to this system are well understood and result in elliptical, parabolic, or hyperbolic orbits, depending on the initial conditions and the total energy of the system.

The stability of an orbit refers to the ability of a trajectory to remain essentially unchanged over time, even in the presence of small perturbations. In the context of the two-body problem, elliptical orbits are considered stable, as small perturbations do not lead to significant changes in the shape or orientation of the orbit. This result follows directly from Kepler's Laws and the differential equations describing the motion of bodies under Newtonian gravitation.

Understanding orbital stability in the two-body problem has numerous practical and theoretical applications. For instance, it is essential for determining artificial satellite trajectories, planning interplanetary space missions, and predicting the motion of asteroids and comets. Although originally derived from empirical observations, Kepler's Laws are now understood as natural consequences of Newton's Law of Universal Gravitation.

In this paper, we presented the classical results of the two-body problem. This model serves as a foundation for studying tidal problems in celestial mechanics and astrophysics. The focus was on bounded orbits, such as circular and elliptical orbits. Studies that consider perturbations to the model, such as the introduction of a third body or additional frictional forces, often employ Delaunay coordinates. This formalism enables the analysis of the model's stability by considering the averaged equations derived from the equations of motion.

However, this was not the primary focus of this work. Our analysis focused on the classical fundamentals and stable elliptical orbits within the context of the two-body problem, without accounting for additional perturbations. Future work could address the effects of body deformations (as Earth is not a perfect sphere) or approach the problem from the perspective of General Relativity (for extreme cases, such as orbits near black holes).

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