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TILING: A NEW INTERPRETATION FOR INTEGER PARTITIONS

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Abstract
This work intends to present a new way to observe the partitions of an integer writing a partition as horizontal tiling created through the representation of a partition as a two-line matrix. The objective of this transformation is to enable a new interpretation of the study of partitions and, from there, study new options and scenarios for the discovery or re-reading of theorems.

Keywords: Integer Partitions, Ramanujan, Tilings, Enumerative Combinatorics

1 Introduction

In [2] Santos, Mondeck and Ribeiro proved a bijection between the number of partitions of n and the number of two line matrices with some specific conditions. Later in [1] Brietzke, Santos and Silva observed another identities that were proven involving matrices under new conditions, combinatorial identities, restricted partitions, and mock theta functions, such as those found in [4], [8], and [9].

Theorem 1.1. *Presented in [2]: There is a bijection between the number of unrestricted partitions of n is and the number of two-line matrices of the form:*

$$\begin{pmatrix} c_1 & c_2 & c_3 & \dots & c_s \\ d_1 & d_2 & d_3 & \dots & d_s \end{pmatrix},$$

where:
$c_s = 0,$
$c_t = c_{t+1} + d_{t+1}, \quad t = 1, 2, \ldots, s-1 \ ,$
$\sum_{t=1}^{s}(c_t + d_t) = n \ .$

Proof. Let M be a two-line matrix as defined in Theorem 1.1. Is easy to see that if we denote each $\lambda_k = c_k + d_k$, $k = 1, 2, \ldots, s$, we will notice that every matrix M can be interpreted as a partition of n since $\lambda_1 + \lambda_2 + \ldots + \lambda_s = n$, where $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_s$. On the other hand, each partition of n can be interpreted as a two-line matrix under the same specified conditions by making $c_s = 0$, $d_s = \lambda_s$ and for $1 \leq t < s-1$, $c_t = \lambda_{t+1}$ and $d_t = \lambda_t - \lambda_{t-1}$. This way, we are guaranteed that $c_t + d_t = \lambda_t$, $1 \leq t \leq s$.
To prove the uniqueness, consider two distinct partitions of n, both with the same number of parts: $\lambda : \lambda_1 + \lambda_2 + \ldots + \lambda_s = n$ and $\omega : \omega_1 + \omega_2 + \ldots + \omega_s = n$ represented by the same matrix $M_\lambda = M_\omega$.
But if the partitions λ and ω are different, should exist any $t = 1, 2, \ldots, s$ such that $\lambda_t \neq \omega_t$, so $c_t + d_t$ in M_λ if different from $c_t + d_t$ in M_ω so both c_t and d_t in these matrices shouldn't be equals, what does it implies $M_\lambda \neq M_\omega$ □

Example 1.2. The partition $\lambda = 6 + 4 + 3 + 2 + 2 + 1$ of $n = 18$ can be represented as the following two-line matrix:

$$\begin{pmatrix} 4 & 3 & 2 & 2 & 1 & 0 \\ 2 & 1 & 1 & 0 & 1 & 1 \end{pmatrix}$$

And this matrix, under conditions established in Theorem 1.1, is uniquely represented.

In [5], Santos and Matte, converted these matrices into reticulated paths by the Cartesian plane, called Path Procedure and in [3], Alegri, Brietzke and Santos, to prove many bijections interpreted these paths as a sequence of distinct odd integers as follows in section 2.

2 The Path-Procedure and The Sequences With Odd Terms

Now known as the bijection between the unrestricted partitions of n and the number of two-line matrices as established in Theorem 1.1, let's build a path through the Cartesian

Plane connecting the origin $(0,0)$ to the point $P=(\sum_{k=1}^{s} d_i, \sum_{k=1}^{s} c_i)$ which lies in the line $x+y=n$. The path consists of shifting d_1 units right, c_1 units up, then d_2 units right, c_2 units up, and so on, ending with d_s units right (because the next step would depend on $c_s=0$). Thus, the path that passes through the points would be created:

$$\begin{aligned}(0,0) &\rightarrow (d_1,0)\rightarrow(d_1,c_1)\rightarrow(d_1+d_2,c_1)\rightarrow(d_1+d_2,c_1+c_2)\rightarrow\ldots\rightarrow\\ &\rightarrow \left(\sum_{k=1}^{s-1} d_s, \sum_{k=1}^{s-1} c_s\right)\rightarrow\left(\sum_{k=1}^{s} d_s, \sum_{k=1}^{s-1} c_s\right).\end{aligned}$$

Remark 2.1. In the original work [5], the path is built in the opposite direction of what was presented above (from P to $(0,0)$). But in this work, we choose to present in reverse order because of the consequences we want to explore.

The Path Procedure will be completed by reflecting the first path described over the line $x+y=n$. This reflection will generate an extension of the first path, from its endpoint and above the line $x+y=n$. This new path will connect point $P=(\sum_{k=1}^{s} d_i, \sum_{k=1}^{s} c_i)$ to point $Q=(\sum_{k=1}^{s} c_i+d_i, \sum_{k=1}^{s} c_i+d_i)$. This reflection consists of shifting d_s units up, c_{s-1} units right, then d_{s-1} units up, c_{s-2} units right, and so on, ending with c_1 units right and d_s units up.

In the end, we will have the path above the line $x+y=n$:

$$\begin{aligned}&\left(\sum_{k=1}^{s} d_s, \sum_{k=1}^{s-1} c_s\right)\rightarrow\left(\sum_{k=1}^{s} d_s, d_s+\sum_{k=1}^{s-1} c_s\right)\rightarrow\left(c_{s-1}+\sum_{k=1}^{s} d_s, d_s+\sum_{k=1}^{s-1} c_s\right)\rightarrow\\ &\rightarrow\left(c_{s-1}+\sum_{k=1}^{s} d_s, d_{s-1}+d_s+\sum_{k=1}^{s-1} c_s\right)\rightarrow\left(c_{s-2}+c_{s-1}+\sum_{k=1}^{s} d_s, d_{s-1}+d_s+\sum_{k=1}^{s-1} c_s\right)\rightarrow\\ &\rightarrow\ldots\rightarrow\left(\sum_{k=1}^{s-1} c_s+\sum_{k=1}^{s} d_s, \sum_{k=1}^{s} d_s+\sum_{k=1}^{s-1} c_s\right)=\left(\sum_{k=1}^{s}(c_i+d_i), \sum_{k=1}^{s}(c_i+d_i)\right)=(n,n)\,.\end{aligned}$$

The Figure 1 exemplifies the Path Procedure applied to the partition $\lambda=3+2+2=7$ and the two-line matrix associated.

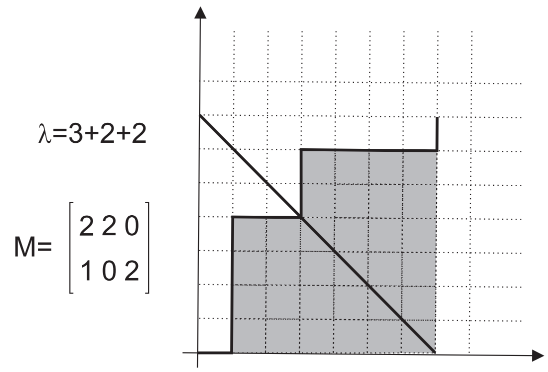

Figure 1: $\lambda = 3 + 2 + 2$ of 7

At that point, we can look at the reflected path and in order associate each shift of length 1 to the ordered sequence odds $1, 3, 5, 7, \ldots$. Among these, we will only consider those that arise from horizontal shifts. The Figures 2, 3 and 4 exemplifies the Path Procedure applied to the partitions $\lambda = 3 + 2 + 2$, $\lambda = 2 + 2 + 2 + 1$ and $\lambda = 1 + 1 + 1 + 1 + 1 + 1 + 1$ all from $n = 7$, the two-line matrices and the sequences of distinct odd numbers associated to each one of them.

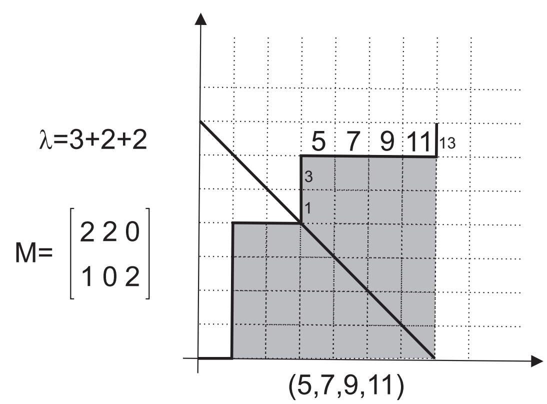

Figure 2: $\lambda = 3 + 2 + 2$ of 7

Once this bijection is completed, we have open space to develop the main theme of this work, which is a new bijection of the unrestricted partitions of n with a tiling under specific conditions.

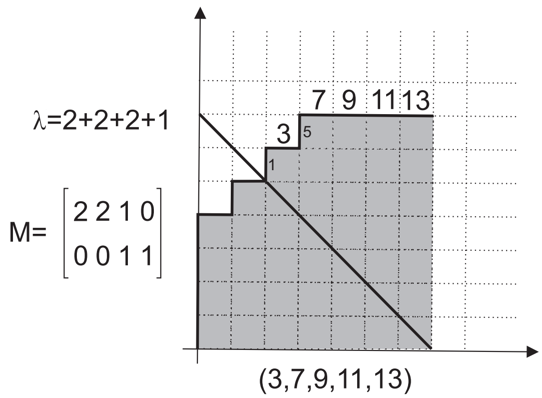

Figure 3: $\lambda = 2 + 2 + 2 + 1$ of 7

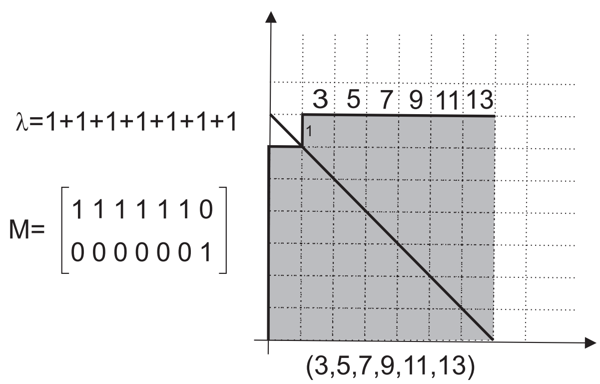

Figure 4: $\lambda = 1 + 1 + 1 + 1 + 1 + 1 + 1$ of 7

3 Tiling

We define a tiling as a sequence of tiles (or squares) juxtaposed horizontally. These tiles can assume two situations: filled and unfilled. Figures 5 and 6 provide an example of the tiles:

Figure 5: Tiling example with length 7

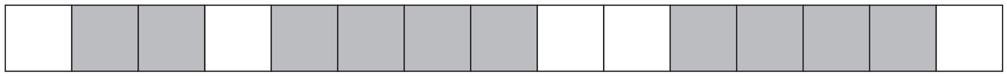

Figure 6: Tiling example with length 15

3.1 Tiling Interpretation

To introduce our main objective of converting each partition into a tiling, let's look at the odd number sequences created in Section 2.

When representing a partition of a integer n as a two line matrix, each entry d_k in the second line results in a sequence of d_k consecutive odd numbers not chosen (because they represents vertical segments after the reflection of the path over $x + y = n$) and each entry c_k in the first line results in sequence of c_k consecutive odd numbers chosen (because they represents horizontal segments in the reflection of the same path). Then, the length of the sequence of odd numbers observed, whether chosen or not, represents n because $\sum_{k=1}^{s}(c_k + d_k) = n$. Example 1 shows this conversion in a practical situation.

Example 3.1. In Section 2, Figure 7 was presented, which represents the partition $\lambda = 3 + 2 + 2$, its matrix and its associated sequence of odd numbers.

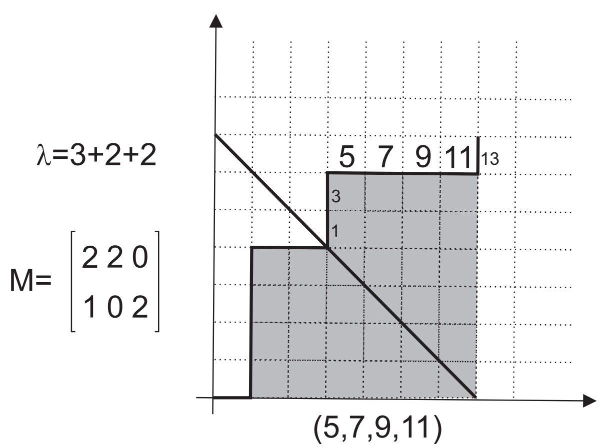

Figure 7: $\lambda = 3 + 2 + 2$ of 7

Analyzing the entries of the two-line matrix in the order $c_3, d_3, c_2, d_2, \ldots$ we obtain the sequence $0, 2, 2, 0, 2, 1$ whose odd-order terms represents sequences of chosen numbers and the even-order terms represents a sequence of not chosen numbers. So we can

find the final sequence relative to this partition:
$0 \rightarrow$ first sequence chosen of odd numbers with length 0 - does not change the sequence;
$2 \rightarrow$ first sequence not chosen of odd numbers with length 2 - at this time it means: $1, 3$;
$2 \rightarrow$ second sequence chosen of odd numbers with length 2 - at this time it means: $5, 7$
$0 \rightarrow$ second sequence not chosen of odd numbers with length 0 - does not change the sequence;
$2 \rightarrow$ third sequence chosen of odd numbers with length 2 - at this time it means: $9, 11$
$1 \rightarrow$ third sequence not chosen of odd numbers with length 1 - at this time it means: 13

Therefore, the sequence of chosen odd numbers is $(5, 7, 9, 11)$ and the sequence of not chosen odd numbers is $(1, 3, 13)$.

Definition 3.2. Each partition $\lambda : \lambda_1 + \lambda_2 + \ldots + \lambda_s = n$ of n will be represented by a n length tiling and each odd number not chosen as described in the Path Procedure Method described in Section 2 will be interpreted as a unfilled tile as well as each odd number chosen will be represented as a filled tile.

Figure 8 shows the tiling equivalent to the partition presented in Example 3.1.

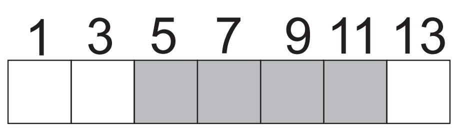

Figure 8: Tiling of $\lambda = 3 + 2 + 2$

Definition 3.3. A tiling of length n, created from a partition of n, after conversion in a two line matrix and the Path Procedure will be called a **valid tiling** and a tiling with no relations to the partitions of n will be called an **invalid tiling**.

Theorem 3.4. *A tiling of length n will be valid if it satisfies two conditions:*

1. *the first tile must be unfilled;*
2. *starting from the second tile, any sequence of filled tiles must have the same length of the number of empty tiles located to its left.*

Proof. We can observe that after reflecting over the line $x + y = n$, the first two sequences of paths to be inserted are the sequences of paths referring to c_s (horizontal) and d_s (vertical), respectively. As $c_s = 0$ and $d_s \neq 0$, the first sequence of odd numbers, with length d_k will always be correspondent to a vertical segment, that is, not chosen odd numbers and this implies in a sequence of empty tiles. Therefore, the first tile of a valid tiling will always be unfilled.

As $c_s = 0$, and each $c_t = c_{t+1} + d_{t+1}$, with $t = 1, 2, \ldots, s-1$, so:

$$c_{s-1} = c_s + d_s = d_s,$$

$$c_{s-2} = c_{s-1} + d_{s-1} = d_s + d_{s-1},$$

$$c_{s-3} = c_{s-2} + d_{s-2} = d_s + d_{s-1} + d_{s-2},$$

$$\vdots$$

$$c_t = \sum_{i=t+1}^{s} (d_i) = d_s + d_{s-1} + \ldots + d_{t+1} \text{ for each } t = 1, 2, \ldots, s-1.$$

So, each entry c_t in the first line is, in fact, the sum of all the entries of the second line located in the columns in the right of column t. From the perspective of odd numbers sequences, the length of each sequence of odd numbers chosen is equal to the sum of the lengths of all odd sequences not chosen up to that step. □

Examples 9, 10 and 11 show some types of tilings and the entries of their reference matrices.

Example 3.5. The Figure 9 represents a valid Tiling of $n = 7$ because the first sequence is formed by unfilled tiles, and the sequences of filled tiles has same length as the sum of the sequences of unfilled tiles up to that moment. The matrix that originated this

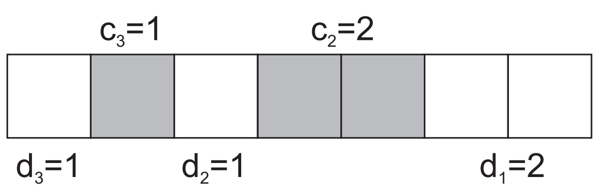

Figure 9: Valid tiling of $n = 7$

tiling is:

$$\begin{pmatrix} 2 & 1 & 0 \\ 2 & 1 & 1 \end{pmatrix}$$

And this matrix represents the partition $\lambda : 4 + 2 + 1 = 7$

Example 3.6. The Figure 10 represents a valid Tiling of $n = 7$ because the first sequence is formed by unfilled tiles, and the sequences of filled tiles has same length as the sum of the sequences of unfilled tiles up to that moment. The matrix that originated

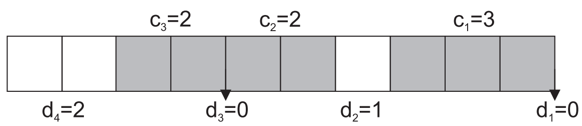

Figure 10: Valid tiling of $n = 10$

this tiling is:

$$\begin{pmatrix} 3 & 2 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{pmatrix}$$

And this matrix represents the partition $\lambda : 3 + 3 + 2 + 2 = 10$

Example 3.7. The Figure 11 represents two invalid Tilings. The first tiling is invalid

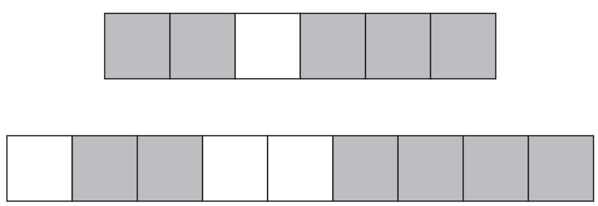

Figure 11: Invalid tilings

because the first tile is filled and the second tiling is invalid because there is a sequence of 4 filled tiles in a moment with 3 unfilled tiles at the left positions.

Thinking about making the analysis of the validity of tilings more practical and visual, we can propose a subtle change in the validity criteria proposed in Theorem 3.4 and we will analyze the validity of a tiling using the criteria proposed in Theorem 3.8.

Theorem 3.8. *A valid tiling, that is, a tiling coming from a partition of n (after the entire process of being converted into a two-line matrix, the path procedure and the sequence of distinct odd numbers) can always be obtained, in a simplified way by the following procedure:*

1. *The first tile will always be unfilled;*
2. *Starting from the tile with position $i \geq 2$, a sequence of consecutive tiles **only** can be filled if its length is a multiple of the number of unfilled tiles in positions j, with $1 \leq j < i$.*

Proof. 1. To prove the first tile: how $c_s = 0$ and $d_s \neq 0$ define the lengths of the first sequences of filled and unfilled tiles respectively then, the first sequence of filled tiles will have length $c_s = 0$, that is, it does not exist and the first sequence of unfilled tiles will have length $d_s \neq 0$. So, at least the first tile of a valid tiling will be unfilled.

2. To prove the length of filled sequences: note that each c_t entry in the first line, $1 \leq t < s$ can be calculated as the sum of the entries in the second line located to its right $d_{t+1} + d_{t+2} + \ldots + d_s$ because:

$$\begin{aligned} c_t &= d_{t+1} + c_{t+1} \\ &= d_{t+1} + d_{t+2} + c_{t+2} \\ &= d_{t+1} + d_{t+2} + d_{t+3} + c_{t+3} \\ &\vdots \\ &= d_{t+1} + d_{t+2} + \ldots + d_s \\ &= \sum_{i=t+1}^{s} d_i. \end{aligned}$$

So, in effect, each sequence of filled tiles is equal to the number of empty tiles to its left. However, when $d_t = 0$, two sequences of filled tiles will appear one in sequence of the other without separation and visually we will have a single sequence whose length is twice the number of empty tiles up to that moment. If in addition to $d_t = 0$, $d_{t+1} = 0$ also, the sequence becomes three times the length of the number of empty tiles and so on. This situation can be seen in Figure 10, when $d_3 = 0$ juxtaposing $c_3 = 2$ and $c_2 = 2$, forming a sequence of 4 filled tiles.

□

4 Final Considerations

This work is an introduction to a significantly larger project, which began in [6], that aims to cover several restricted and unrestricted partitions of integers from the perspective of tiling. In other papers already submitted to publication, we interpret each tiling as a number written in base 2, completing a bijection between the unrestricted partitions of any n with an odd number. The standard behavior of some Slater [7] and Ramanujan identities and their predictable interpretations from the perspective of tiling were also observed.

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