

RELATIONSHIPS ON MULTIDIMENSIONAL EXTENSIONS OF *ONE-ZERO* NUMBERS

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Abstract

In this paper, we define multidimensional version of *One-Zero* sequence and look for analogous characterizations with the classic uni-dimensional sequence. Here, we would like to proceed by analogy with the uni-dimensional case in order to obtain similar results for bi-dimensional and k -dimensional for any non-negative integer k . We prove, in particular, results for a bi-dimensional case, and with mathematical support we extend our study concerning the k -dimensional identities for the *One-zero* sequence. As a consequence of such properties, we exhibit the Binet Formula associated with k -dimensional identities of this sequence. Furthermore, we exhibit arithmetic properties for the bi-dimensional sequence, and we study the partial sums of terms in higher dimensions.

Keywords: Binet's Formula; *One-Zero* sequence; k -dimensional sequence.

1 Introduction

For a non-negative integer n , consider the *One-Zero* sequences $\{U_n\}_{n \geq 0}$ that are the terms of the set $\{0, 1, 101, 10101, 1010101, \dots\}$, where each term satisfies the recursive

relation $U_{n+1} = 100U_n + 1$, for all $n \geq 0$ and $U_0 = 0$, with the id-number *A094028* in OEIS [21]. For more details about *One-Zero* sequence, see, for instance, the works [3, 7, 9, 19]. In recent study [8], it was demonstrated that this sequence also satisfies a Horadam-like recurrence relation:

$$U_{n+2} = 101U_{n+1} - 100U_n . \quad (1.1)$$

with the initial terms $U_0 = 0$ and $U_1 = 1$.

Recall that the recurrence Equation (1.1) is associated with characteristic equation related to the *One-Zero* sequence that is $\lambda^2 - 101\lambda + 100 = 0$. This equation is a particular case of the characteristic equation of a Horadam-type sequence. The distinct roots of this equation are $\lambda_1 = 100$ and $\lambda_2 = 1$.

In Equation (2.3) in [8], we can find non-recursive forms to obtain *One-Zero* numbers, which are given by

$$U_n = \frac{10^{2n} - 1}{99} \quad \text{for all } n \geq 0. \quad (1.2)$$

The Equation (1.2) is the Binet-like formula for *One-Zero* sequence.

The sequences usually consist of consecutive numbers that follow a certain recurrence rule and initial values. In this paper, we define and introduce the k -dimensional *One-Zero* sequences for any non-negative integer k , with a particular focus on the bidimensional case. Our objective is to develop a theoretical framework to generalize these sequences to arbitrary dimensions. Using the Binet formula, we derive several identities for the k -dimensional or multidimensional sequence.

In the literature, there are several studies exploring the k -dimensional forms of Horadam-like sequences for $k \geq 0$. For instance, the authors in [15] investigate bidimensional and tridimensional identities for Fibonacci numbers in their complex form, while in [16] it was introduced and analyzed Gaussian Fibonacci numbers along with their bidimensional recurrence relations. In [14], it was studied the n -dimensional forms associated with the Mersenne sequence. In [17], it was provided an overview of the Leonardo sequence and examined the bidimensional recurrence relations derived from its one-dimensional model. Likewise, [2, 13] explores the bidimensional and tridimensional recurrence relations of the Narayana sequence, extending its unidimensional recursive structure. In [4, 5], the authors present bidimensional extensions of balancing, Lucas-balancing, cobalancing, and Lucas-cobalancing numbers, along with an analysis of the properties of these novel bidimensional sequences. Recently, [6] introduces and analyzes the k -dimensional relations of Repunit sequence. These and other related studies have inspired further research into bidimensional extensions of additional numerical sequences, building upon their unidimensional foundations.

The objective of this paper is to investigate some connections between the k -dimensional *One-Zero* sequence and the classical *One-Zero* sequence. The article is structured as follows: the next section presents the background and auxiliary results related to unidimensional or ordinary *One-Zero* numbers, which provide the foundation for the subsequent analysis. In Section 3, we introduce the bidimensional version of the *One-Zero* sequence and exhibit some of its properties; in subsection 3.1 it will be presented some of the properties satisfied by the bidimensional *One-Zero* sequence, in particular the some classical identities; while in subsection 3.2 it will be focused on the partial sums of its terms. In Section 4, we define and study the tridimensional *One-Zero* sequence and present some results related to the bidimensional *One-Zero* sequence and again present the definition of multidimensional *One-Zero* sequence and their consequence. Finally, the definition and properties are extended to k -dimensional *One-Zero* sequences.

In this research work, we will show how to associate particular points in the plane with a bidimensional sequence, by analogy to the unidimensional case. To being more precise, this analogy also leads to define *One-Zero* bidimensional sequences. This paper addresses the recurrence relations that model the k -dimensional case of this sequence, providing a comprehensive analysis of its structure and properties. Our main results are given in Section 3.

2 Background and Preliminaries Results

In this section, we present some results related to the one-dimensional version of the *One-Zero* sequence. Furthermore, we present an auxiliary result that will be useful in more advanced stages of the investigation. So, these auxiliary results will play a crucial role in establishing new identities for the k -dimensional forms of this sequence.

The first result is about the difference and the sum of two *One-Zero* terms, and the proofs are immediate by the use of Binet's formula. We omit the proofs.

Proposition 2.1. *For all non-negative integers n and k . If $n \geq k$ then the identities hold:*

- (a) $U_n - U_k = 10^{2k}U_{n-k}$,
 - (b) $U_n + U_k = U_{n-k} \cdot 10^{2k} + 2U_k$,
- where $\{U_n\}_{n \geq 0}$ is the *One-zero* sequence.

Specifically, when $n = k + 1$, we obtain:

Corollary 2.2. For all non-negative integers k , we have

- (a) $U_{k+1} - U_k = 10^{2k}$,

- (b) $U_{k+1} + U_k = 10^{2k} + 2U_k$,
 where $\{U_n\}_{n \geq 0}$ is the *One-zero* sequence.

Also by using the Binet formula and in a similar approach to the one used in Proposition 2.1, the two following results are valid.

Proposition 2.3. *Let n be any non-negative integer, then*

$$U_{n+1} - 100U_{n-1} = 10^{2n} + 1 ,$$

where $\{U_n\}_{n \geq 0}$ is the *One-zero* sequence.

Proposition 2.4. *Let m , r , and s be arbitrary non-negative integers. Then*

$$U_{m+r} + U_{m+s} - (U_m + U_{m+r+s}) = -99 \cdot 10^{2m} U_r U_s ,$$

where $\{U_n\}_{n \geq 0}$ is the *One-zero* sequence.

Now, we need to recall three results stated in [8]. The next result is known in the mathematical literature as the Tagiuri-Vajda identity.

Lemma 2.5. [8, Theorem 3.4] *Let n be any non-negative integer, then*

$$U_{2n} = 99U_n^2 - 2U_n ,$$

where $\{U_n\}_{n \geq 0}$ is the *One-zero* sequence.

Lemma 2.6. [8, Theorem 3.8] *Let m, r, s be any non-negative integer numbers. We have*

$$U_{m+r}U_{n+s} - U_mU_{m+r+s} = 10^{2m}U_rU_s ,$$

where $\{U_n\}_{n \geq 0}$ is the *One-Zero* sequence.

To conclude this section we consider the summation of the first $n + 1$ terms of the *One-Zero* sequence . We denote the partial sums by $\sum_{k=0}^n U_k = U_0 + U_1 + U_2 + \cdots + U_n$, where $\{U_n\}_{n \geq 0}$ denotes the *One-Zero* sequence.

Lemma 2.7. [8, Proposition 4.1] *Let $\{U_n\}_{n \geq 0}$ be the *One-Zero* sequence and n be a non-negative integer. Then:*

$$(a) \sum_{k=0}^n U_k = \frac{U_{n+1} - (n+1)}{99} ,$$

$$(b) \sum_{k=0}^n U_{2k} = \frac{(10^{2(n+1)} + 1)U_{n+1}}{9999} - \frac{n+1}{99} ,$$

$$(c) \sum_{k=0}^n U_{2k+1} = \frac{10^2(10^{2(n+1)} + 1)U_{n+1}}{9999} - \frac{n+1}{99} .$$

The *One-Zero* sequence is a Horadam-type sequence from the *Gersenne* sequence family, the partial sum expressed here is a particular case of Proposition 12, recently published in [20].

3 Bidimensional *One-Zero* Sequence

According to the terminology of [10], the Gaussian numbers, denoted by (a, b) , are all complex numbers of the form $(a, b) = a + bi$, for all integer numbers a and b . We find several studies in the literature that explore the extension of unidimensional identities of various other sequences to higher dimensions with some additional conditions for some Horadam-type sequences, as can be seen in [1, 11, 12, 18, 22].

In the beginning, we explore the extension of the *One-Zero* sequence to a complex or bidimensional domain. Now we define the bidimensional *One-Zero* sequence and provide some properties of this new sequence.

Definition 3.1. For all integers $m \geq 0$ and $n \geq 0$, the bidimensional *One-Zero* sequences $\{U_{(m,n)}\}_{m,n \geq 0}$ are defined recursively by

$$U_{(m,n)} =: \begin{cases} 101U_{(m-1,n)} - 100U_{(m-2,n)}, & \text{for all } n \text{ and } m \geq 2 \\ 101U_{(m,n-1)} - 100U_{(m,n-2)}, & \text{for all } m \text{ and } n \geq 2, \end{cases} \quad (3.1)$$

with initial terms $U_{(0,0)} = 0$, $U_{(1,0)} = 1$, $U_{(0,1)} = i$, $U_{(1,1)} = 1 + i$, and where $i^2 = -1$ is the imaginary unit.

Definition 3.1 is valid in the sense that the general term $U_{(m,n)}$ is independent of the path we use. For instance, to find $U_{(2,2)}$, we see that

$$\begin{cases} 101U_{(1,2)} - 100U_{(0,2)}, & \text{path (a)} \\ 101U_{(2,1)} - 100U_{(2,0)}, & \text{path (b)} \end{cases} .$$

- Let us consider path (a): we have

$$\begin{aligned} U_{(1,2)} &= 101U_{(1,1)} - 100U_{(1,0)} = 111(1 + i) - 100 \cdot 1 \\ &= 101 + 101i - 100 = 1 + 101i, \end{aligned}$$

and

$$U_{(0,2)} = 101U_{(0,1)} - 100U_{(0,0)} = 101i - 100 \cdot 0 = 101i,$$

hence,

$$\begin{aligned} U_{(2,2)} &= 101U_{(1,2)} - 100U_{(0,2)} = 101(1 + 101i) - 100(101i) \\ &= 101 + 10201i - 10100i = 101 + 101i. \end{aligned}$$

- Consider path (b): we obtain

$$U_{(2,1)} = 101U_{(1,1)} - 100U_{(0,1)} = 101 + i,$$

and

$$U_{(2,0)} = 101U_{(1,0)} - 100U_{(0,0)} = 101,$$

hence,

$$\begin{aligned} U_{(2,2)} &= 101U_{(2,1)} - 100U_{(2,0)} = 101(101 + i) - 100(101) \\ &= 10201 + 101i - 10100 = 101 + 101i, \end{aligned}$$

which is the same as the value we found for $U_{(2,2)}$ where we have used the path (a).

By means of direct calculations, we obtain three branches of the bidimensional *One-Zero* sequences as an illustrative approach:

$$\{U_{(m,0)}\}_{m \geq 0} = \{0, 1, 101, 10101, 1010101, 101010101, \dots\},$$

$$\{U_{(0,n)}\}_{n \geq 0} = \{0, i, 101i, 10101i, 1010101i, 101010101i, \dots\},$$

and

$$\{U_{(n,n)}\}_{n \geq 0} = \{0, 1 + i, 101 + 101i, 10101 + 10101i, 1010101 + 1010101i, \dots\}.$$

3.1 Some Properties

In the following, a brief overview of the properties of bidimensional *One-Zero* numbers are provided.

Proposition 3.2. *Let m and n denote arbitrary non-negative integers. The following properties hold:*

$$U_{(m,0)} = U_m; \quad (3.2)$$

$$U_{(0,n)} = U_n i; \quad (3.3)$$

$$U_{(m,1)} = U_{m+1} + i; \quad (3.4)$$

$$U_{(1,1)} = 1 + U_1 i; \quad (3.5)$$

where $\{U_{(m,n)}\}_{m,n \geq 0}$ is the bidimensional One-Zero sequence, $\{U_n\}_{n \geq 0}$ is the One-Zero sequence, and $i^2 = -1$.

Demonstração. For Equation (3.2), the proof is carried out by induction on m . For $m = 0$ and $m = 1$, we have: $U_{(0,0)} = 0 = U_0$ and $U_{(1,0)} = 1 = U_1$. Thus, the property is true for $m = 0$ and $m = 1$. Assume that the property is true for all values less than or equal to m , that is, $U_{(k,0)} = U_k$ for $k \leq m$. We need to prove that the property holds for $m + 1$. By (3.1), we have:

$$U_{(m+1,0)} = 101U_{(m,0)} - 100U_{(m-1,0)}.$$

Using the inductive hypothesis, we have:

$$U_{(m+1,0)} = 101U_m - 100U_{m-1}.$$

But by Equation (1.1), we conclude

$$U_{(m+1,0)} = U_{m+1}.$$

Therefore, by the principle of mathematical induction, property from Equation (3.2) is true for all non-negative integers m .

In a similar way, the proof of Equation (3.3) can be done by induction on n . Again, in order to prove Equation (3.4), we proceed by induction on m . And finally, the proof of Equation (3.5) can also be done by induction on n . $\square$

The following result establishes a connection between the bidimensional One-Zero sequence and the classical One-Zero sequence. It is our main result.

Theorem 3.3. *For non-negative integers m and n , the bidimensional One-Zero numbers are described as follows:*

$$U_{(m,n)} = U_m + U_n i,$$

where $\{U_{(m,n)}\}_{m,n \geq 0}$ is the bidimensional One-Zero sequence, $\{U_n\}_{n \geq 0}$ is the One-Zero sequence, and $i^2 = -1$.

Demonstração. We start by applying induction to n , with m held constant. Let us consider $n = 0$. Note that by initial conditions $U_{(0,0)} = 0 = 0 + 0i = U_0 + U_0i$ and $U_{(1,0)} = 1 = 1 + 0i = U_1 + U_0i$. For $m \geq 2$ held constant, we have

$$\begin{aligned} U_{(m,0)} &= U_m = U_m + 0i \\ &= U_m + U_0i, \end{aligned}$$

which is true by Equation (3.2), as $U_0 = 0$. Suppose the statement is true for all $k \leq n$. Let us prove for $n + 1$. By (3.1) we have:

$$U_{(m,n+1)} = 101U_{(m,n)} - 100U_{(m,n-1)}.$$

Now, by the induction hypothesis, we get:

$$\begin{aligned} U_{(m,n+1)} &= 101(U_m + U_ni) - 100(U_m + U_{n-1}i) \\ &= U_m + (101U_n - 100U_{n-1})i, \end{aligned}$$

following Equation (1.1) that

$$U_{(m,n+1)} = U_m + U_{n+1}i,$$

which verifies the result.

In a similar way, let us now fix n and perform the induction on m . □

By the previous result and Equation (1.2), we can obtain the Binet-like formula for the bidimensional *One-Zero* sequence.

Corollary 3.4 (Binet's formula). For non-negative integers n and m , we have

$$U_{(m,n)} = \frac{10^{2m} - 1}{99} + \frac{10^{2n} - 1}{99}i,$$

or equivalently

$$U_{(m,n)} = \frac{10^{2m} + 10^{2n}i - (1 + i)}{99},$$

where $\{U_{(m,n)}\}_{m,n \geq 0}$ is the bidimensional *One-Zero* sequence, and $i^2 = -1$.

By combining Proposition 2.1 and Theorem 3.3, we obtain the following result.

Proposition 3.5. *Let us consider non-negative integers m , n , and k , with $m \geq k$ and $n \geq k$. Let $\{U_{(m,n)}\}_{m,n \geq 0}$ be the bidimensional *One-Zero* sequence. Then the identities are verified:*

- (a) $U_{(m,n)} - U_{(k,n)} = 10^{2k}U_{m-k}$,
 - (b) $U_{(m,n)} - U_{(m,k)} = 10^{2k}U_{n-k}i$,
 - (c) $U_{(m,n)} + U_{(k,n)} = (U_{(m-k)} \cdot 10^{2k} + 2U_k) + U_n i$,
 - (d) $U_{(m,n)} + U_{(m,k)} = U_m + (U_{(m-k)} \cdot 10^{2k} + 2U_k)i$,
- where $\{U_n\}_{n \geq 0}$ is the *One-Zero* sequence, and $i^2 = -1$.

A direct consequence of the Proposition 3.5 is the next result.

Corollary 3.6. Let us consider non-negative integers m and n . Let $\{U_{(m,n)}\}_{m,n \geq 0}$ be the bidimensional *One-Zero* sequence. Then the identities are validated:

- (a) $U_{(m+1,n)} - U_{(m,n)} = 10^{2m}$,
 - (b) $U_{(m,n+1)} - U_{(m,n)} = 10^{2m}i$,
 - (c) $U_{(m+1,n)} + U_{(m,n)} = (10^{2m} + U_m) + U_n i$,
 - (d) $U_{(m,n+1)} + U_{(m,n)} = U_m + (10^{2n} + 2U_n)i$,
- where $\{U_n\}_{n \geq 0}$ is the *One-Zero* sequence, and $i^2 = -1$.

The following result follows from the combination of Proposition 2.3 and Theorem 3.3.

Proposition 3.7. For non-negative integers m and n and for the bidimensional *One-Zero* sequence $\{U_{(m,n)}\}_{m,n \geq 0}$, the identities are verified:

- (a) $U_{(m+1,n)} - 100U_{(m-1,n)} = (10^{2m} + 1) - 99U_{n \geq 0}i$,
 - (b) $U_{(m,n+1)} - 100U_{(m,n-1)} = -99U_m + (10^{2n} + 1)i$,
- where $\{U_m\}_m$ is the *One-Zero* sequence, and $i^2 = -1$.

Combining Lemma 2.5 and Theorem 3.3 yields the following result.

Proposition 3.8. For non-negative integers m and n and for the bidimensional *One-Zero* sequence $\{U_{(m,n)}\}_{m,n \geq 0}$, the identities are verified:

- (a) $U_{(2m,n)} - 2U_{(m,n)} = 99U_m^2 - U_{n \geq 0}i$,
 - (b) $U_{(m,2n)} - 2U_{(m,n)} = -U_m + 99U_n^2i$,
- where $\{U_n\}_{n \geq 0}$ is the *One-Zero* sequence, and $i^2 = -1$.

The classical Tagiuri-Vajda identity of the unidimensional *One-Zero* sequence is extended to two dimensions in the following results.

To begin with, we will consider n which is fixed in the second coordinate of the indexes and which varies in the first coordinate.

Theorem 3.9 (First Tagiuri-Vajda's identity). *Let m, n, r , and s be arbitrary non-negative integers, the following identity is verified*

$$U_{(m+r,n)}U_{(m+s,n)} - U_{(m,n)}U_{(m+r+s,n)} = 10^{2m}U_rU_s(1 - 99U_ni) , \quad (3.6)$$

where $\{U_{(m,n)}\}_{m,n \geq 0}$ is the bidimensional One-Zero sequence, $\{U_n\}_{n \geq 0}$ is the One-Zero sequence, and $i^2 = -1$.

Demonstração. By Theorem 3.3 we have

$$\begin{aligned} & U_{(m+r,n)}U_{(m+s,n)} - U_{(m,n)}U_{(m+r+s,n)} \\ &= (U_{m+r} + U_ni)(U_{m+s} + U_ni) - (U_m + U_ni)(U_{m+r+s} + U_ni) \\ &= (U_{m+r}U_{m+s} - U_mU_{m+r+s}) + (U_{m+r} + U_{m+s} - U_m - U_{m+r+s})U_ni . \end{aligned}$$

Applying Proposition 2.4 and Lemma 2.6 simultaneously, we get

$$\begin{aligned} & U_{(m+r,n)}U_{(m+s,n)} - U_{(m,n)}U_{(m+r+s,n)} = \\ &= 10^{2m}U_rU_s - 99 \cdot 10^{2m}U_rU_sU_ni , \end{aligned}$$

and we have the validity of the result. $\square$

From Tagiuri-Vajda's identity, we get the results establishing d'Ocagne's identity, Catalan's identity, and Cassini's identity for the bidimensional *One-Zero* sequence $\{U_{(m,n)}\}_{m,n \geq 0}$.

Proposition 3.10 (First d'Ocagne's identity). *For non-negative integers m, n and t with $t \geq m$, it holds:*

$$U_{(t,n)}U_{(m+1,n)} - U_{(m,n)}U_{(t+1,n)} = 10^{2m}U_{t-m}(1 - 99U_ni) .$$

where $\{U_{(m,n)}\}_{m,n \geq 0}$ is the bidimensional One-Zero sequence, $\{U_n\}_{n \geq 0}$ is the One-Zero sequence, and $i^2 = -1$.

Demonstração. Taking $r = t - m$ and $s = 1$ in Equation (3.6) gives

$$U_{(t,n)}U_{(m+1,n)} - U_{(m,n)}U_{(t+1,n)} = 10^{2m}U_{t-m}U_1(1 - 99U_ni) ,$$

as $U_1 = 1$, and this completes the proof. $\square$

We have the Catalan identity in a similar way to Proposition 3.10.

Proposition 3.11 (First Catalan's identity). *For non-negative integers t and s with $t \geq s$, it holds:*

$$U_{(t,n)}^2 - U_{(t-s,n)}U_{(t+s,n)} = 10^{2(t-s)}U_s^2(1 - 99U_n i) .$$

where $\{U_{(m,n)}\}_{m,n \geq 0}$ is the bidimensional *One-Zero* sequence, $\{U_n\}_{n \geq 0}$ is the *One-Zero* sequence, and $i^2 = -1$.

Demonstração. Using $r = s$ and $m + s = t$ in the Equation (3.6), we have the result. $\square$

At the expense of the above result, we obtain the Cassini identity.

Corollary 3.12 (First Cassini's identity). Let $t \geq 1$ be any integers, then

$$U_{(t,n)}^2 - U_{(t-1,n)}U_{(t+1,n)} = 10^{2(t-1)}(1 - 99U_n i) ,$$

where $\{U_{(m,n)}\}_{m,n \geq 0}$ is the bidimensional *One-Zero* sequence, $\{U_n\}_{n \geq 0}$ is the *One-Zero* sequence, and $i^2 = -1$.

Demonstração. Taking $s = 1$ in Proposition 3.11. $\square$

A similar result can be obtained by considering the variation of the indexes in the second coordinate. We omit the proof of the next result in the interest of brevity.

Theorem 3.13 (Second Tagiuri-Vajda's identity). *Let m , n , r , and s be arbitrary non-negative integers, the following identity is verified*

$$U_{(m,n+r)}U_{(m,n+s)} - U_{(m,n)}U_{(m,n+r+s)} = 10^{2n}U_rU_s(1 - 99U_m i) ,$$

where $\{U_{(m,n)}\}_{m,n \geq 0}$ is the bidimensional *One-Zero* sequence, $\{U_n\}_{n \geq 0}$ is the *One-Zero* sequence, and $i^2 = -1$.

Proposition 3.14 (Second d'Ocagne's identity). *For non-negative integers m, n and t with $t \geq m$, it holds:*

$$U_{(m,t)}U_{(m,n+1)} - U_{(m,n)}U_{(m,t+1)} = 10^{2n}U_{t-n}(1 - 99U_m i) .$$

where $\{U_{(m,n)}\}_{m,n \geq 0}$ is the bidimensional *One-Zero* sequence, $\{U_n\}_{n \geq 0}$ is the *One-Zero* sequence, and $i^2 = -1$.

Proposition 3.15 (Second Catalan's identity). *For non-negative integers t and s with $t \geq s$, it holds:*

$$U_{(m,t)}^2 - U_{(m,t-s)}U_{(m,t+s)} = 10^{2(t-s)}U_s^2(1 - 99U_m i) .$$

where $\{U_{(m,n)}\}_{m,n \geq 0}$ is the bidimensional *One-Zero* sequence, $\{U_n\}_{n \geq 0}$ is the *One-Zero* sequence, and $i^2 = -1$.

Corollary 3.16 (Second Cassini's identity). Let $t \geq 1$ be any integers, then

$$U_{(m,t)}^2 - U_{(m,t-1)}U_{(m,t+1)} = 10^{2(t-1)}(1 - 99U_m i),$$

where $\{U_{(m,n)}\}_{m,n \geq 0}$ is the bidimensional *One-Zero* sequence, $\{U_n\}_{n \geq 0}$ is the *One-Zero* sequence, and $i^2 = -1$.

3.2 Some Partial Sum

In this section, we present the results concerning the partial sums of the first $k + 1$ terms of the bidimensional *One-Zero* sequence $\{U_{(m,n)}\}_{m,n \geq 0}$. We begin by examining the sequence of partial sums in the first coordinate of the indexes, given by

$$\sum_{m=0}^k U_{(m,n)} = U_{(0,n)} + U_{(1,n)} + U_{(2,n)} + \cdots + U_{(k,n)}$$

for all integers n , where $\{U_{(m,n)}\}_{m,n \geq 0}$ represents the bidimensional *One-Zero* sequence.

Proposition 3.17. For non-negative integers m and n , let $\{U_{(m,n)}\}_{m,n \geq 0}$ be the bidimensional *One-Zero* sequence and k a non-negative integer. Then:

$$\begin{aligned} (a) \quad \sum_{m=0}^k U_{(m,n)} &= \frac{U_{k+1} - (k+1)}{99} + (k+1)U_n i, \\ (b) \quad \sum_{m=0}^k U_{(2m,n)} &= \frac{(10^{2(k+1)} + 1)U_{k+1}}{9999} - \frac{(k+1)}{99} + (k+1)U_n i, \\ (c) \quad \sum_{m=0}^k U_{(2m+1,n)} &= \frac{10^2(10^{2(k+1)} + 1)U_{k+1}}{9999} - \frac{(k+1)}{99} + (k+1)U_n i, \end{aligned}$$

where $\{U_n\}_{n \geq 0}$ is the *One-Zero* sequence.

Demonstração.

(a) By Lemma 2.7, item (a) and Theorem 3.3 we have,

$$\begin{aligned} \sum_{m=0}^k U_{(m,n)} &= \sum_{m=0}^k (U_m + U_n i) = \sum_{m=0}^k U_m + \sum_{m=0}^k U_n i \\ &= \frac{U_{k+1} - (k+1)}{99} + (k+1)U_n i, \end{aligned}$$

which is the end of the proof.

(b) By Lemma 2.7, item (b) and Theorem 3.3 we have,

$$\begin{aligned}\sum_{m=0}^k U_{(2m,n)} &= \sum_{m=0}^k (U_{2m} + U_n i) = \sum_{m=0}^k U_{2m} + \sum_{m=0}^k U_n i \\ &= \frac{(10^{2(k+1)} + 1)U_{k+1}}{9999} - \frac{(k+1)}{99} + (k+1)U_n i,\end{aligned}$$

which proves the result.

(c) Similarly, by Lemma 2.7, item (c) and Theorem 3.3 we have:

$$\begin{aligned}\sum_{m=0}^k U_{(2m+1,n)} &= \sum_{m=0}^k (U_{2m+1} + U_n i) = \sum_{m=0}^k U_{2m+1} + \sum_{m=0}^k U_n i \\ &= \frac{10^2(10^{2(k+1)} + 1)U_{k+1}}{9999} - \frac{(k+1)}{99} + (k+1)U_n i,\end{aligned}$$

and this completes the proof. $\square$

Consider now the sequence of alternating partial sums given by

$$\sum_{m=0}^k (-1)^m U_{(m,n)} = U_{(0,n)} - U_{(1,n)} + U_{(2,n)} - U_{(3,n)} + \cdots + (-1)^k U_{(k,n)}$$

for all integers n , where $\{U_{(m,n)}\}_{m,n \geq 0}$ denotes the bidimensional *One-Zero* sequence.

Proposition 3.18. *For non-negative integers m , n , and k , let $\{U_{(m,n)}\}_{m,n \geq 0}$ be the bidimensional *One-Zero* sequence. Then:*

$$\begin{aligned}(a) \quad \sum_{m=0}^{2k+1} (-1)^m U_{(m,n)} &= -\frac{(10^{2(k+1)} + 1)U_{k+1}}{101}, \\ (b) \quad \sum_{m=0}^{2k} (-1)^m U_{(m,n)} &= \frac{10^{2k}U_{k+1}}{101} - \frac{102}{9999} + U_n i,\end{aligned}$$

where $\{U_n\}_{n \geq 0}$ is the *One-Zero* sequence.

Demonstração. Combining items (b) and (c) of the Proposition 3.17. $\square$

Now, in the second coordinate, we examine the sequence of partial sums of the $n+1$ first terms, given by

$$\sum_{n=0}^k U_{(m,n)} = U_{(m,0)} + U_{(m,1)} + \cdots + U_{(m,k)}$$

for $m \geq 0$, where $\{U_{(m,n)}\}_{m,n \geq 0}$ represents the bidimensional *One-Zero* sequence.

Similarly to Proposition 3.17, we have the following result. In the interest of brevity, we will omit the proof.

Proposition 3.19. *For non-negative integers m , n , and k , let $\{U_{(m,n)}\}_{m,n \geq 0}$ be the bidimensional *One-Zero* sequence. Then:*

$$\begin{aligned}
 (a) \quad & \sum_{n=0}^k U_{(m,n)} = (k+1)U_m + \frac{U_{k+1} - (k+1)}{99}i, \\
 (b) \quad & \sum_{n=0}^k U_{(m,2n)} = (k+1)U_m + \left(\frac{(10^{2(k+1)} + 1)U_{k+1}}{9999} - \frac{(k+1)}{99} \right)i, \\
 (c) \quad & \sum_{m=0}^k U_{(2m+1,n)} = (k+1)U_m + \left(\frac{10^2(10^{2(k+1)} + 1)U_{k+1}}{9999} - \frac{(k+1)}{99} \right)i,
 \end{aligned}$$

where $\{U_n\}_{n \geq 0}$ is the *One-Zero* sequence.

Combining items (b) and (c) of the Proposition 3.19 we have the next result.

Proposition 3.20. *For non-negative integers m , n , and k , let $\{U_{(m,n)}\}_{m,n \geq 0}$ be the bidimensional *One-Zero* sequence. Then:*

$$\begin{aligned}
 (a) \quad & \sum_{n=0}^{2k+1} (-1)^m U_{(m,n)} = -\frac{(10^{2(k+1)} + 1)U_{k+1}}{101}i, \\
 (b) \quad & \sum_{n=0}^{2k} (-1)^m U_{(m,n)} = U_m + \left(\frac{10^{2k}U_{k+1}}{101} - \frac{102}{9999} \right)i,
 \end{aligned}$$

where $\{U_n\}_{n \geq 0}$ is the *One-Zero* sequence.

4 Multidimensional *One-Zero* Sequence

Initially, in this section, we define and study a tridimensional *One-Zero* sequence and we provide an enumeration of its properties. We use the previous characterization for bidimensional *One-Zero* sequence to obtain extension of results in tridimensional or k -dimensional *One-Zero* sequence.

The following definition is offered as a point of departure:

Definition 4.1. For integers $n_1 \geq 0$, $n_2 \geq 0$ and $n_3 \geq 0$, the tridimensional *One-Zero* sequence $\{U_{(n_1, n_2, n_3)}\}_{n_1, n_2, n_3 \geq 0}$ is defined recursively by

$$U_{(n_1, n_2, n_3)} = \begin{cases} 101U_{(n_1-1, n_2, n_3)} - 100U_{(n_1-2, n_2, n_3)} & \text{for all } n_2, n_3, \text{ and } n_1 \geq 2, \\ 101U_{(n_1, n_2-1, n_3)} - 100U_{(n_1, n_2-2, n_3)} & \text{for all } n_1, n_3, \text{ and } n_2 \geq 2, \\ 101U_{(n_1, n_2, n_3-1)} - 100U_{(n_1, n_2, n_3-2)} & \text{for all } n_1, n_2, \text{ and } n_3 \geq 2, \end{cases}$$

with initial terms $U_{(0,0,0)} = 0$, $U_{(1,0,0)} = 1$, $U_{(0,1,0)} = i$, $U_{(0,0,1)} = j$, $U_{(1,1,0)} = 1 + i$, $U_{(1,0,1)} = 1 + j$, $U_{(0,1,1)} = i + j$, $U_{(1,1,1)} = 1 + i + j$, where $i^2 = j^2 = -1$ are imaginary units.

Definition 4.1 holds because $U_{(n_1, n_2, n_3)}$ remains unchanged regardless of the computation path chosen. To illustrate this, we present three different approaches to derive $U_{(2,2,2)}$. Thus, in calculating $U_{(2,2,2)}$, we observe that:

$$U_{(2,2,2)} =: \begin{cases} 101U_{(1,2,2)} - 100U_{(0,2,2)}, & \text{path (a);} \\ 101U_{(2,1,2)} - 100U_{(2,0,2)}, & \text{path (b);} \\ 101U_{(2,2,1)} - 100U_{(2,2,0)}, & \text{path (c).} \end{cases} \quad (4.1)$$

To start, we conduct and summarize some preliminary calculations:

$$\begin{aligned} U_{(0,0,2)} &= 101U_{(0,0,1)} - 100U_{(0,0,0)} = 101j - 100 \cdot 0 = 101j; \\ U_{(0,2,0)} &= 101U_{(0,1,0)} - 100U_{(0,0,0)} = 101i - 100 \cdot 0 = 101i; \\ U_{(2,0,0)} &= 101U_{(1,0,0)} - 100U_{(0,0,0)} = 101 \cdot 1 - 100 \cdot 0 = 101; \\ U_{(0,1,2)} &= 101U_{(0,1,1)} - 100U_{(0,1,0)} = 101(i + j) - 100 \cdot i = i + 101j; \\ U_{(1,0,2)} &= 101U_{(1,0,1)} - 100U_{(1,0,0)} = 101(1 + i) - 100 \cdot 1 = 1 + 101j; \\ U_{(0,2,1)} &= 101U_{(0,1,1)} - 100U_{(0,0,1)} = 101(i + j) - 100 \cdot j = 101i + j; \\ U_{(1,2,0)} &= 101U_{(1,1,0)} - 100U_{(1,0,0)} = 101(1 + i) - 100 \cdot 1 = 1 + 101j; \\ U_{(2,0,1)} &= 101U_{(1,0,1)} - 100U_{(0,0,1)} = 101(1 + i) - 100 \cdot j = 101 + 101i - 100j; \\ U_{(2,1,0)} &= 101U_{(1,1,0)} - 100U_{(0,1,0)} = 101(1 + i) - 100 \cdot i = 101 + i; \\ U_{(1,1,2)} &= 101U_{(1,1,1)} - 100U_{(1,1,0)} = 101(1 + i + j) - 100(1 + i) = 1 + i + 101j; \\ U_{(1,2,1)} &= 101U_{(1,1,1)} - 100U_{(1,0,1)} = 101(1 + i + j) - 100(1 + j) = 1 + 101i + j; \\ U_{(2,1,1)} &= 101U_{(1,1,1)} - 100U_{(0,1,1)} = 101(1 + i + j) - 100(i + j) = 101 + i + j. \end{aligned}$$

- Consider path (a): we have

$$\begin{aligned}
U_{(1,2,2)} &= 101U_{(1,2,1)} - 100U_{(1,2,0)} \\
&= 101(1 + 101i + j) - 100 \cdot (1 + 101i) \\
&= 1 + 101i + 101j,
\end{aligned}$$

and

$$\begin{aligned}
U_{(0,2,2)} &= 101U_{(0,2,1)} - 100U_{(0,2,0)} \\
&= 101 \cdot (101i + j) - 100 \cdot 101i = 101i + 101j,
\end{aligned}$$

hence,

$$\begin{aligned}
U_{(2,2,2)} &= 101U_{(1,2,2)} - 100U_{(0,2,2)} \\
&= 101(1 + 101i + 101j) - 100(101i + 101j) \\
&= 101 + 101i + 101j.
\end{aligned}$$

- Consider path (b): we obtain

$$\begin{aligned}
U_{(2,1,2)} &= 101U_{(1,1,2)} - 100U_{(0,1,2)} \\
&= 101 + i + 101j,
\end{aligned}$$

and

$$\begin{aligned}
U_{(2,0,2)} &= 101U_{(1,0,2)} - 100U_{(0,0,2)} \\
&= 101 + 101j,
\end{aligned}$$

hence,

$$\begin{aligned}
U_{(2,2,2)} &= 101U_{(1,2,2)} - 100U_{(0,2,2)} \\
&= 101 + 101i + 101j.
\end{aligned}$$

- Consider path (c): we have the following

$$\begin{aligned}
U_{(2,2,1)} &= 11U_{(2,1,1)} - 10U_{(2,0,1)} \\
&= 101 + 101i + j,
\end{aligned}$$

and

$$\begin{aligned} U_{(2,0,2)} &= 101U_{(1,0,2)} - 100U_{(0,0,2)} \\ &= 101 + 101j, \end{aligned}$$

hence,

$$\begin{aligned} U_{(2,2,2)} &= 101U_{(1,2,2)} - 100U_{(0,2,2)} \\ &= 101 + 101i + 101j. \end{aligned}$$

this result matches the value obtained for $U_{(2,2,2)}$ when calculated using either path (a) or path (b).

We will illustrate, as an example, one branch of the tridimensional *One-Zero* sequence:

$$\{U_{(0,0,n)}\}_{n \geq 0} = \{0, 1j, 101j, 10101j, 1010101j, \dots\}.$$

In a similar way to what we have done in Section 3, we will have similar results in this section, but here we will only present the statement, without the respective proofs.

Proposition 4.2. *Let n_1 , n_2 , and n_3 denote arbitrary non-negative integers. The following properties hold:*

$$\begin{aligned} U_{(n_1,0,0)} &= U_{n_1} , \\ U_{(0,n_2,0)} &= U_{n_2} i , \\ U_{(0,0,n_3)} &= U_{n_3} j , \\ U_{(n_1,1,0)} &= U_{n_1} + i , \\ U_{(n_1,0,1)} &= U_{n_1} + j , \\ U_{(n_1,1,1)} &= U_{n_1} + i + j , \\ U_{(1,n_2,1)} &= 1 + U_{n_2} i , \\ U_{(1,n_2,1)} &= 1 + U_{n_2} i + j , \\ U_{(1,0,n_3)} &= 1 + U_{n_3} j , \\ U_{(1,1,n_3)} &= 1 + i + U_{n_3} j , \end{aligned}$$

where $\{U_{(n_1,n_2,n_3)}\}_{n_1,n_2,n_3 \geq 0}$ is the tridimensional *One-Zero* sequence, $\{U_n\}_{n \geq 0}$ is the *One-Zero* sequence, and $i^2 = j^2 = -1$.

The following result connect the tridimensional *One-Zero* numbers to the unidimensional *One-Zero* numbers.

Proposition 4.3. *For non-negative integers n_1 , n_2 , and n_3 , the tridimensional One-Zero numbers are described as follows:*

$$U_{(n_1, n_2, n_3)} = U_{n_1} + U_{n_2}i + U_{n_3}j,$$

where $\{U_{(n_1, n_2, n_3)}\}_{n_1, n_2, n_3 \geq 0}$ is the tridimensional One-Zero sequence, $\{U_n\}_{n \geq 0}$ is the One-Zero sequence, and $i^2 = j^2 = -1$.

Thus, using the previous result and Binet's formula from Equation (1.2), we derive the Binet formula for tridimensional One-Zero sequence, as follows.

Corollary 4.4. [Binet's formula] For non-negative integers n_1 , n_2 and n_3 , we have

$$U_{(n_1, n_2, n_3)} = \frac{10^{2n_1} - 1}{99} + \frac{10^{2n_2} - 1}{99}i + \frac{10^{2n_3} - 1}{99}j,$$

or equivalently

$$U_{(n_1, n_2, n_3)} = \frac{10^{2n_1} + 10^{2n_2}i + 10^{2n_3}j - (1 + i + j)}{99},$$

where $\{U_{(n_1, n_2, n_3)}\}_{n_1, n_2, n_3 \geq 0}$ is the tridimensional One-Zero sequence, and $i^2 = j^2 = -1$.

The next results are an extension of the results of the tridimensional One-zero, so we will only present the statement with respect to the third component of the indexes.

Proposition 4.5. *For all non-negative integers n_1 , n_2 , n_3 , and k , with $n_3 \geq k$. Let $\{U_{(n_1, n_2, n_3)}\}_{n_1, n_2, n_3 \geq 0}$ be the tridimensional One-Zero sequence, then the following identities are verified:*

- (a) $U_{(n_1, n_2, n_3)} - U_{(n_1, n_2, k)} = U_{n_3-k}10^{2k}j$,
- (b) $U_{(n_1, n_2, n_3+1)} - U_{(n_1, n_2, n_3-1)} = -99U_{n_1} - 99U_{n_2}i + (10^{2n_3} + 1)j$,
- (c) $U_{(n_1, n_2, 2n_3)} - U_{(n_1, n_2, n_3)} = -U_{n_1} - U_{n_2}i + 99U_{n_3}^2j$, where $\{U_m\}_m$ is the One-Zero sequence, and $i^2 = j^2 = -1$.

The proof is done similarly to Propositions 3.5, 3.7, and 3.8.

We are now in a position to define the generalized case. Now let us extend the Definition 3.1 and so we introduce the multidimensional or k -dimensional One-zero sequence for all integers $k \geq 2$,

Definition 4.6. For integers $n_1 \geq 0, n_2 \geq 0, \dots, n_k \geq 0$ and $k \geq 2$, the k -dimensional *One-Zero* sequence $\{U_{(n_1, n_2, \dots, n_k)}\}_{n_1, n_2, \dots, n_k \geq 0}$ is defined recursively by

$$\begin{cases} U_{(n_1+1, n_2, \dots, n_k)} = 11U_{(n_1, n_2, \dots, n_k)} - 10U_{(n_1-1, n_2, \dots, n_k)}, \\ U_{(n_1, n_2+1, \dots, n_k)} = 11U_{(n_1, n_2, \dots, n_k)} - 10U_{(n_1, n_2-1, \dots, n_k)}, \\ \vdots \\ U_{(n_1, n_2, \dots, n_k+1)} = 11U_{(n_1, n_2, \dots, n_k)} - 10U_{(n_1, n_2, \dots, n_k-1)}, \end{cases} \quad (4.2)$$

with initial terms: $U_{(0,0,0,\dots,0)} = 0$, $U_{(1,0,0,\dots,0)} = 1$, $U_{(0,1,0,\dots,0)} = i_2$, $U_{(0,0,1,0,\dots,0)} = i_3$, $U_{(1,1,0,\dots,0)} = 1 + i_2$, $U_{(1,0,1,0,\dots,0)} = 1 + i_3$, $U_{(1,1,1,0,\dots,0)} = 1 + i_2 + i_3, \dots$, $U_{(1,1,1,\dots,1)} = 1 + i_2 + i_3 + \dots + i_k$, where $i_2^2 = i_3^2 = \dots = i_k^2 = -1$ are imaginary units.

Combining the Definition 4.6 and the results established in Proposition 4.3 and Corollary 4.4, we state the following key result:

Proposition 4.7. *For any k -tuple $(n_1, n_2, \dots, n_k)$ of non-negative integers, the general term of the set $\{U_{(n_1, n_2, \dots, n_k)}\}_{(n_1, n_2, \dots, n_k)}$ is given by*

$$U_{(n_1, n_2, \dots, n_k)} = U_{n_1} + U_{n_2}i_2 + \dots + U_{n_k}i_k,$$

or equivalently:

$$U_{(n_1, n_2, \dots, n_k)} = \frac{10^{2n_1} + 10^{2n_2}i_2 + \dots + 10^{2n_k}i_k - (1 + i_2 + \dots + i_k)}{9},$$

where $\{U_{(n_1, n_2, \dots, n_k)}\}_{n_1, n_2, \dots, n_k \geq 0}$ is the k -dimensional *One-Zero* sequence, $\{U_{n_j}\}_{n_j \geq 0}$ ($j = 1, 2, \dots, k$) is the *One-Zero* sequence, and $i_2^2 = i_3^2 = \dots = i_k^2 = -1$.

According to the reference [8], the results are also valid for negative indexes. Specifically, we have:

$$U_{(-n_1, -n_2, -n_3, \dots, -n_k)} = U_{-n_1} + U_{-n_2}i_2 + \dots + U_{-n_k}i_k.$$

Thus, all the properties verified in this research can also be extended to the case where the indexes are negative.

5 Conclusion

Number sequences constitute an important subject area in mathematics, and the uni-dimensional model of the *One-Zero* sequence served as a scope for the introduction

of imaginary units, facilitating the development of bidimensional and tridimensional recurrences and paving the way for generalizations to k -dimensional case. This complexification of the usual or unidimensional *One-Zero* sequence was achieved by extending the recurrence model and the initial terms, which resulted in its generalization to a k -dimensional structure. This work presents a comprehensive analysis of the complexification process of the *One-Zero* sequence and its relationship with Horadam-type sequences. The complexification was obtained by incorporating the imaginary units and extending it to an algebraic representation in higher dimensions. Consequently, the bidimensional recurrence relations of the *One-Zero* sequence were established, along with several identities in their complex forms. These contributions offer valuable tools regarding the structural complexity of these numbers through a rigorous mathematical analysis of their recurrence equation. This provided new results and sometimes extensions of some existing results in the literature. These results highlight the versatility of *One-Zero* numbers and their generalizations, providing a basis for further exploration of their properties in theoretical and applied contexts.

Author Contributions

E.A.C., D.C.S. and P.M.M.C.C. contributed equally to this work. All authors have read and agreed to the published version of the manuscript.

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