

ON GENERALIZED (s,t) -JACOBSTHAL AND THEIR ASSOCIATED NUMBERS

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Handled by M. Alegri

Abstract

In this work, we have studied the (s, t) -Jacobsthal-type sequence, a two-parameter extension of the classical Jacobsthal sequence. A most important contribution of the work is to establish some identities satisfied by the (s, t) -Jacobsthal-type sequence; these include extended versions of classical identities such as the Vajda identity, d'Ocagne's identity, Catalan's identity, Cassini–Simson's identity, and Gelin–Cesàro's identity. Using a new approach, we have obtained new results that greatly complement existing results and several works in the literature. In addition, we also present some results for s -Jacobsthal, providing a specification for $t = 1$ in the sequence (s, t) -Jacobsthal-type sequence.

Keywords: Jacobsthal numbers. (s, t) -Jacobsthal numbers. Tagiuri–Vajda's identity.

1 Introduction

We found many works discussing generalization of Jacobsthal numbers, whose recurrence relation became known through the following recurrence $J_{n+1} = J_n + 2J_{n-1}$ with initial terms given by $J_0 = 0$ and $J_1 = 1$. Using this recurrence, we obtain the following list $\{J_n\}_{n \geq 0}$:

$$0, 1, 1, 3, 5, 11, 21, 43, 85, 171, 341, \dots, \quad (1.1)$$

The Jacobsthal sequence is listed as A001045 in the OEIS [17]. Furthermore, we may consider the recurrence $j_{n+1} = j_n + 2j_{n-1}$ with initial terms $j_0 = 2$ and $j_1 = 1$, which determine the set of numbers $\{j_n\}_{n \geq 0}$:

$$2, 1, 5, 7, 17, 31, 65, 127, 257, \dots, \quad (1.2)$$

and is known as the Jacobsthal–Lucas sequence, listed as A014551 in the OEIS [17].

We have come across a long tradition of works with an interest in the generalization processes of Jacobsthal numbers, for instance [1, 2, 3, 4, 5, 6, 8, 7, 9, 10, 11, 12, 13, 14, 15, 16, 18, 19] and references therein. In these terms, among the diversity of approaches, highlights the generalization introduced, primarily, in [1, 2] and [11, 15], that informs our motivation for the following definitions.

We start presenting a one-parameter generalization of Jacobsthal-type numbers.

Definition 1.1. For any integer $n \geq 0$ and any real number $s > 0$.

- (a) [2, Equation (2)] The set of s -Jacobsthal $\{J_n^{(s)}\}_{n \geq 0}$ is defined by the following relation

$$J_n^{(s)} = \frac{s^n - (-1)^n}{s + 1}. \quad (1.3)$$

- (b) [11, Equation (8)] The set of s -Jacobsthal–Lucas $\{j_n^{(s)}\}_{n \geq 0}$ is defined by the following relation

$$j_n^{(s)} = s^n + (-1)^n. \quad (1.4)$$

The Table 1, and Table 2 present, respectively, a few terms of s -Jacobsthal and s -Jacobsthal–Lucas.

Table 1: Some particular terms of s -Jacobsthal numbers $\{J_n^{(s)}\}_{n \geq 0}$.

$J_0^{(s)}$	$J_1^{(s)}$	$J_2^{(s)}$	$J_3^{(s)}$	$J_4^{(s)}$	$J_5^{(s)}$	$\dots$
0	1	$s - 1$	$s^2 - s + 1$	$s^3 - s^2 + s - 1$	$s^4 - s^3 + s^2 - s + 1$	$\dots$

Source: prepared by authors.

Table 2: Some particular terms of s -Jacobsthal–Lucas numbers $\{J_n^{(s)}\}_{n \geq 0}$.

$j_0^{(s)}$	$j_1^{(s)}$	$j_2^{(s)}$	$j_3^{(s)}$	$j_4^{(s)}$	$j_5^{(s)}$	$j_6^{(s)}$	$j_7^{(s)}$	$\dots$
2	$s - 1$	$s^2 + 1$	$s^3 - 1$	$s^4 + 1$	$s^5 - 1$	$s^6 + 1$	$s^7 - 1$	$\dots$

Source: prepared by authors.

It can be seen that by replacing $s = 2$ in the Equation (1.3), we obtain the Jacobsthal numbers (1.1). Whereas, replacing $s = 2$ in the Equation (1.4), we get the Jacobsthal–Lucas numbers (1.2).

Now we present a two-parameter generalization of Jacobsthal-type numbers.

Definition 1.2. For any integer $n \geq 0$ and any positive real numbers s and t ,

- (a) [1, pag. 38] The set of (s, t) -Jacobsthal $\{J_n^{(s,t)}\}_{n \geq 0}$ is defined by the following relation

$$J_n^{(s,t)} = \frac{s^n - (-t)^n}{s + t}; \quad (1.5)$$

- (b) [15, pag. 69] The set of (s, t) -Jacobsthal–Lucas $\{j_n^{(s,t)}\}_{n \geq 0}$ is defined by the following relation

$$j_n^{(s,t)} = s^n + (-t)^n. \quad (1.6)$$

Tables 3 and 4 display several initial terms of the (s, t) -Jacobsthal and (s, t) -Jacobsthal–Lucas sequences, respectively.

Table 3: Some particular terms of (s, t) -Jacobsthal numbers $\{J_n^{(s,t)}\}_{n \geq 0}$.

$J_0^{(s,t)}$	$J_1^{(s,t)}$	$J_2^{(s,t)}$	$J_3^{(s,t)}$	$J_4^{(s,t)}$	$J_5^{(s,t)}$	$\dots$
0	1	$s - t$	$s^2 - st + t^2$	$s^3 - ts^2 + t^2s - t^3$	$s^4 - ts^3 + t^2s^2 - t^3s + t^4$	$\dots$

Source: prepared by authors.

Table 4: Some particular terms of (s, t) -Jacobsthal–Lucas numbers $\{j_n^{(s,t)}\}_{n \geq 0}$.

$j_0^{(s,t)}$	$j_1^{(s,t)}$	$j_2^{(s,t)}$	$j_3^{(s,t)}$	$j_4^{(s,t)}$	$j_5^{(s,t)}$	$j_6^{(s,t)}$	$j_7^{(s,t)}$	$\dots$
2	$s - t$	$s^2 + t^2$	$s^3 - t^3$	$s^4 + t^4$	$s^5 - t^5$	$s^6 + t^6$	$s^7 - t^7$	$\dots$

Source: prepared by authors.

Note that, when $t = 1$ in Equation (1.5) gives us the s -Jacobsthal (1.3). Similarly, replacing $t = 1$ in Equation (1.6) we obtain the s -Jacobsthal–Lucas (1.4).

The (s, t) -Jacobsthal $\{J_n^{(s,t)}\}_{n \geq 0}$ can be extended to negative subscripts. Indeed, we can consider, for all $n \geq 0$,

$$\begin{aligned} J_{-n}^{(s,t)} &= \frac{s^{-n} - (-t)^{-n}}{s + t} = \frac{1}{(-st)^n} \left(\frac{(-t)^n - s^n}{s + t} \right) \\ &= \frac{1}{(-1)^{n+1}(st)^n} \left(\frac{s^n - (-t)^n}{s + t} \right) = \frac{1}{(-1)^{n+1}(st)^n} J_n^{(s,t)}, \end{aligned}$$

so,

$$J_{-n}^{(s,t)} = \frac{(-1)^{n+1}}{(st)^n} J_n^{(s,t)}. \quad (1.7)$$

Analogously,

$$j_{-n}^{(s,t)} = \frac{(-1)^n}{(st)^n} j_n^{(s,t)}.$$

In what follows, the term (s, t) -Jacobsthal-type sequence will be used to refer to either the (s, t) -Jacobsthal or (s, t) -Jacobsthal–Lucas sequences. In this article, we will point out certain new properties that involve generalizations of classical identities related to Jacobsthal-type sequence. In addition to this introduction, the work consists of two more sections. In Section 2 we properly carried out studies of the two-parameter Jacobsthal-type numbers presented in recurrence relations and some consequences, some relationships between (s, t) -Jacobsthal or (s, t) -Jacobsthal–Lucas, as well as some classical identities, such as the Tagiuri-Vajda identity for the (s, t) -Jacobsthal-type, followed by some derivations and consequences. While in Section 3, we present the study of one-parameter Jacobsthal-type sequence in which in most of the results only $t = 1$ is specified for something asserted for (s, t) -Jacobsthal-type sequence.

2 Two-parameter Jacobsthal-type sequence

In this section we study some properties and identities for (s, t) -Jacobsthal numbers and (s, t) -Jacobsthal–Lucas numbers.

2.1 Recurrence relations and some consequences

By looking Tables 3 and 4, we can verify the following properties derived directly from Definition 1.2, that is, we note that Equation (1.5) and (1.6) are the Binet formula for a second-order linear recurrence sequence of the form given by

Lemma 2.1. *Let $\{J_n^{(s,t)}\}_{n \geq 0}$ be the (s, t) -Jacobsthal and $\{j_n^{(s,t)}\}_{n \geq 0}$ be the (s, t) -Jacobsthal-Lucas sequence. For all integers $n \geq 0$ and any positive real numbers s, t the following identities are true:*

- (a) $J_{n+2}^{(s,t)} = (s - t)J_{n+1}^{(s,t)} + tsJ_n^{(s,t)}$ with initial terms $J_0^{(s,t)} = 0$ and $J_1^{(s,t)} = 1$,
 (b) $j_{n+2}^{(s,t)} = (s - t)j_{n+1}^{(s,t)} + tsj_n^{(s,t)}$ with initial terms $j_0^{(s,t)} = 2$ and $j_1^{(s,t)} = s - t$.

The Lemma 2.1 can be found in some works, for instance in [13], Equations (3) and (24), respectively.

We obtain two non-homogeneous recurrence relations for each sequence in the following result.

Proposition 2.2. *For any integer $n \geq 0$ and any positive real numbers s, t , the following identities hold:*

$$J_{n+1}^{(s,t)} = sJ_n^{(s,t)} + (-t)^n, \quad (2.1)$$

$$j_{n+1}^{(s,t)} = sj_n^{(s,t)} - (s + t)(-t)^n, \quad (2.2)$$

$$J_{n+1}^{(s,t)} = -tJ_n^{(s,t)} + s^n, \quad (2.3)$$

$$j_{n+1}^{(s,t)} = -tj_n^{(s,t)} + (s + t)s^n, \quad (2.4)$$

where $\{J_n^{(s,t)}\}_{n \geq 0}$ and $\{j_n^{(s,t)}\}_{n \geq 0}$ are, respectively, the (s, t) -Jacobsthal sequence and the (s, t) -Jacobsthal-Lucas sequence.

Proof. For the proof of (2.1), see Theorem 2 in [1], and for the proof of (2.2) see Theorem 2.2 in [15].

By Equation (1.5) we have

$$\begin{aligned} J_{n+1}^{(s,t)} + t \cdot J_n^{(s,t)} &= \frac{s^{n+1} - (-t)^{n+1}}{s + t} + t \cdot \frac{s^n - (-t)^n}{s + t} \\ &= \frac{s \cdot s^n - (-t)(-t)^n + ts^n - t(-t)^n}{s + t} \\ &= \frac{(s + t) \cdot s^n}{s + t}, \end{aligned}$$

which concludes (2.3).

Now, by Equation (1.6) we get

$$\begin{aligned} j_{n+1}^{(s,t)} + t \cdot j_n^{(s,t)} &= s^{n+1} + (-t)^{n+1} + t \cdot s^n + t(-t)^n \\ &= s \cdot s^n + (-t)(-t)^n + t \cdot s^n + t(-t)^n \\ &= (s + t) \cdot s^n, \end{aligned}$$

as stated in (2.4). □

The next result follows directly from Proposition 2.2.

Theorem 2.3. *Let $\{J_n^{(s,t)}\}_{n \geq 0}$ and $\{j_n^{(s,t)}\}_{n \geq 0}$ be, respectively, the (s, t) -Jacobsthal and the (s, t) -Jacobsthal-Lucas sequences. For all integers $n \geq 1$ and any positive real numbers s, t , the following identities are valid:*

$$(t+1)(s-1) \sum_{i=0}^n J_i^{(s,t)} = (t+1)J_{n+1}^{(s,t)} + (-t)^{n+1} - 1, \quad (2.5)$$

$$(t+1)(s-1) \sum_{i=0}^n j_i^{(s,t)} = (t+1)(j_{n+1}^{(s,t)} - 2) - (s+t)((-t)^{n+1} - 1). \quad (2.6)$$

Proof. By Equation (2.1), we have

$$\begin{aligned} \sum_{i=0}^{n+1} J_i^{(s,t)} &= J_0^{(s,t)} + \sum_{i=1}^{n+1} J_i^{(s,t)} \\ &= J_0^{(s,t)} + s \sum_{i=0}^n J_i^{(s,t)} + \sum_{i=0}^n (-t)^i. \end{aligned} \quad (2.7)$$

On the other hand

$$\sum_{i=0}^{n+1} J_i^{(s,t)} = \left(\sum_{i=0}^n J_i^{(s,t)} \right) + J_{n+1}^{(s,t)}. \quad (2.8)$$

Since $J_0^{(s,t)} = 0$, $\sum_{i=0}^n (-t)^i = \frac{1-(-t)^{n+1}}{t+1}$, and as $t \neq -1$, and combining Equations (2.7) and (2.8), we have

$$(s-1) \sum_{i=0}^n J_i^{(s,t)} = J_{n+1}^{(s,t)} - \frac{1-(-t)^{n+1}}{t+1},$$

which concludes the proof of Equation 2.5.

In a similar way, by Equation (2.2), we have

$$\sum_{i=0}^{n+1} j_i^{(s,t)} = j_0^{(s,t)} + s \sum_{i=0}^n j_i^{(s,t)} - (s+t) \sum_{i=0}^n (-t)^i, \quad (2.9)$$

and

$$\sum_{i=0}^{n+1} j_i^{(s,t)} = \left(\sum_{i=0}^n j_i^{(s,t)} \right) + j_{n+1}^{(s,t)}. \quad (2.10)$$

Since $j_0^{(s,t)} = 2$, $\sum_{i=0}^n (-t)^i = \frac{1-(-t)^{n+1}}{t+1}$, and as $t \neq -1$, and combining Equations (2.9) and (2.10), we have

$$(s-1) \sum_{i=0}^n j_i^{(s,t)} = j_{n+1}^{(s,t)} - 2 + (s+t) \frac{1-(-t)^{n+1}}{t+1},$$

which is the conclusion of the proof of Equation (2.6). $\square$

2.2 Relationship between (s,t)-Jacobsthal-type sequences

In this section, we present relations between the (s, t) -Jacobsthal $\{J_n^{(s,t)}\}_{n \geq 0}$ and (s, t) -Jacobsthal-Lucas $\{j_n^{(s,t)}\}_{n \geq 0}$ sequences.

As a consequence of the Definitions 1.1 and 1.2, we obtain the following results.

Proposition 2.4. *For any integer $n \geq 1$ and any positive real numbers s, t , the following identities hold:*

$$J_{n+1}^{(s,t)} - (ts)J_{n-1}^{(s,t)} = (s-t)J_n^{(s,t)}, \quad (2.11)$$

$$J_{n+1}^{(s,t)} + (st)J_{n-1}^{(s,t)} = j_n^{(s,t)}, \quad (2.12)$$

$$(s-t)J_n^{(s,t)} + j_n^{(s,t)} = 2J_{n+1}^{(s,t)}, \quad (2.13)$$

$$j_{n+1}^{(s,t)} + (st)j_{n-1}^{(s,t)} = (s+t)^2 J_n^{(s,t)}, \quad (2.14)$$

where $J_n^{(s,t)}$ and $j_n^{(s,t)}$ represent the n -th terms of the (s, t) -Jacobsthal and (s, t) -Jacobsthal-Lucas sequences, respectively.

Proof. The Equations (2.11), (2.12) and (2.14) follow directly from Equations (1.5) and (1.6). Equation (2.13) follows by summing the equations (2.11) and (2.12). $\square$

In a similar way, we obtain the following identities.

Proposition 2.5. *For any $n \geq 1$ and any positive real numbers s, t , the following identities are valid:*

$$(s+t)^2 J_n^{(s,t)} J_{n+1}^{(s,t)} = j_{2n+1}^{(s,t)} - (-1)^n (s-t)(st)^n, \quad (2.15)$$

$$(s+t)^2 (J_n^{(s,t)})^2 = j_{2n}^{(s,t)} - 2(-1)^n (st)^n, \quad (2.16)$$

$$J_{n+1}^{(s,t)} j_n^{(s,t)} = J_{2n+1}^{(s,t)} + (-1)^n (st)^n, \quad (2.17)$$

$$J_n^{(s,t)} j_{n+1}^{(s,t)} = J_{2n+1}^{(s,t)} - (-1)^n (st)^n, \quad (2.18)$$

$$j_n^{(s,t)} j_{n+1}^{(s,t)} = j_{2n+1}^{(s,t)} + (-1)^n (s-t)(st)^n, \quad (2.19)$$

where $J_n^{(s,t)}$ and $j_n^{(s,t)}$ represent the n -th terms of the (s,t) -Jacobsthal and (s,t) -Jacobsthal–Lucas sequences, respectively.

Proof. By use of the Equations (1.5) and (1.6), we have

$$\begin{aligned} J_n^{(s,t)} j_{n+1}^{(s,t)} &= \frac{s^n - (-t)^n}{s+t} (s^{n+1} + (-t)^{n+1}) \\ &= \frac{1}{s+t} [s^{2n+1} + (-t)(-st)^n - (-st)^n s - (-t)^{2n+1}] \\ &= \frac{1}{s+t} [s^{2n+1} - (-st)^n (s+t) - (-t)^{2n+1}] \\ &= \frac{1}{s+t} [s^{2n+1} - (-t)^{2n+1}] - (-st)^n = J_{2n+1}^{(s,t)} - (-1)^n (st)^n, \end{aligned}$$

and then we have obtained Equation (2.18). Equations (2.15), (2.16), (2.17), and (2.19) are obtained in a similar manner. $\square$

We write $\{g_n^{(s,t)}\}_{n \geq 0}$ to denote a generalized (s,t) -Jacobsthal sequence, that is, a sequence defined by the following recursive expression $g_{n+2}^{(s,t)} = (s-t)g_{n+1}^{(s,t)} + (st)g_n^{(s,t)}$, for $n \geq 0$, with initial terms $g_0^{(s,t)} = a$ and $g_1^{(s,t)} = b$ where a and b are fixed real constants.

The next result is an extension for (s,t) -Jacobsthal sequence of the result found in [20, Equation (8)]. Specifically, the result shows that for any Fibonacci-type sequence $\{g_n\}_{n \geq 0}$ holds

$$g_{n+m} = f_{m-1}g_n + f_m g_{n+1},$$

where $\{f_n\}_{n \geq 0}$ is the ordinary Fibonacci sequence.

Theorem 2.6. *For any integers $m, n \geq 0$, and for any positive real numbers s, t , the generalized (s,t) -Jacobsthal sequence $\{g_n^{(s,t)}\}_{n \geq 0}$ satisfies the following identity*

$$g_{n+m}^{(s,t)} = (st)J_{m-1}^{(s,t)} \cdot g_n^{(s,t)} + J_m^{(s,t)} \cdot g_{n+1}^{(s,t)}, \quad (2.20)$$

where $\{J_n^{(s,t)}\}_{n \geq 0}$ is the (s,t) -Jacobsthal sequence.

Proof. By induction on m . For $m = 0$ and $m = 1$, the identity holds since $J_{-1}^{(s,t)} = \frac{1}{st}$, $J_0^{(s,t)} = 0$ and $J_1^{(s,t)} = 1$. Now, suppose that (2.20) is verified for every value less than or equal to m , for some $m \geq 2$. Then,

$$\begin{aligned} g_{n+(m+1)}^{(s,t)} &= (s-t)g_{n+m}^{(s,t)} + (st)g_{n+m-1}^{(s,t)} \\ &= (s-t) \left[(st)J_{m-1}^{(s,t)}g_n^{(s,t)} + J_m^{(s,t)}g_{n+1}^{(s,t)} \right] + (st) \left[(st)J_{m-2}^{(s,t)}g_n^{(s,t)} + J_{m-1}^{(s,t)}g_{n+1}^{(s,t)} \right] \\ &= (st) \left[(s-t)J_{m-1}^{(s,t)} + (st)J_{m-2}^{(s,t)} \right] g_n^{(s,t)} + \left[(s-t)J_m^{(s,t)} + (st)J_{m-1}^{(s,t)} \right] g_{n+1}^{(s,t)} \\ &= (st)J_m^{(s,t)}g_n^{(s,t)} + J_{m+1}^{(s,t)}g_{n+1}^{(s,t)}. \end{aligned}$$

That is, the identity holds for every integer m . $\square$

As a direct consequence, we have the following results.

Corollary 2.7. For any integers $m, n \geq 0$, and for any positive real numbers s, t , the generalized (s, t) -Jacobsthal sequence $\{g_n^{(s,t)}\}_{n \geq 0}$ satisfies the following identity

$$(-1)^m (st)^m g_{n-m}^{(s,t)} = J_{m+1}^{(s,t)} \cdot g_n^{(s,t)} - J_m^{(s,t)} \cdot g_{n+1}^{(s,t)}, \quad (2.21)$$

where $\{J_n^{(s,t)}\}_{n \geq 0}$ is the (s, t) -Jacobsthal sequence.

Proof. Now, replacing m with $-m$ in (2.20) and using (1.7), we obtain

$$\begin{aligned} g_{n-m}^{(s,t)} &= (st)J_{-m-1}^{(s,t)} \cdot g_n^{(s,t)} + J_{-m}^{(s,t)} \cdot g_{n+1}^{(s,t)} \\ &= \frac{(-1)^m}{(st)^m} J_{m+1}^{(s,t)} \cdot g_n^{(s,t)} - \frac{(-1)^m}{(st)^m} J_m^{(s,t)} \cdot g_{n+1}^{(s,t)}. \end{aligned}$$

Therefore, by multiplying $(-1)^m (st)^m$ on both sides, we obtain the result. $\square$

Corollary 2.8. For any integers n, m such that $m + n \geq 0$ and for any positive real numbers s, t , the following identities hold:

$$J_{n+m}^{(s,t)} = (st)J_{m-1}^{(s,t)} \cdot J_n^{(s,t)} + J_m^{(s,t)} \cdot J_{n+1}^{(s,t)}, \quad (2.22)$$

$$j_{n+m}^{(s,t)} = (st)J_{m-1}^{(s,t)} \cdot j_n^{(s,t)} + J_m^{(s,t)} \cdot j_{n+1}^{(s,t)}, \quad (2.23)$$

where $\{J_n^{(s,t)}\}_{n \geq 0}$ is the (s, t) -Jacobsthal and $\{j_n^{(s,t)}\}_{n \geq 0}$ is the (s, t) -Jacobsthal-Lucas sequences.

Proof. Take $g_n^{(s,t)} = J_n^{(s,t)}$ and $g_n^{(s,t)} = j_n^{(s,t)}$ in Equation (2.20). $\square$

Corollary 2.9. For any integers n, m such that $n + m \geq 0$, $n - m \geq 0$, and for any positive real numbers s, t , the following identities hold:

$$2J_{n+m}^{(s,t)} = J_m^{(s,t)} \cdot j_n^{(s,t)} + J_n^{(s,t)} \cdot j_m^{(s,t)}, \quad (2.24)$$

$$2(-1)^m (st)^m J_{n-m}^{(s,t)} = J_n^{(s,t)} \cdot j_m^{(s,t)} - J_m^{(s,t)} \cdot j_n^{(s,t)}, \quad (2.25)$$

$$2j_{n+m}^{(s,t)} = (s+t)^2 J_n^{(s,t)} \cdot J_m^{(s,t)} + j_n^{(s,t)} \cdot j_m^{(s,t)}, \quad (2.26)$$

$$2(-1)^m (st)^m j_{n-m}^{(s,t)} = j_m^{(s,t)} \cdot j_n^{(s,t)} - (s+t)^2 J_m^{(s,t)} \cdot J_n^{(s,t)}, \quad (2.27)$$

where $\{J_n^{(s,t)}\}_{n \geq 0}$ is the (s, t) -Jacobsthal and $\{j_n^{(s,t)}\}_{n \geq 0}$ is the (s, t) -Jacobsthal-Lucas sequences.

Proof. By taking $g_i = J_i^{(s,t)}$ and $g_i = j_i^{(s,t)}$ in (2.21), summing with (2.22) and using Equations (2.11), (2.12), (2.13) and (2.14), we get the Equations (2.24), (2.25), (2.26) and (2.27). $\square$

In the following, we present an auxiliary result that involves two generalized (s, t) -Jacobsthal sequences of a result for the Fibonacci sequence that can be found in [20], Equation (18).

Theorem 2.10. *For any generalized (s, t) -Jacobsthal sequences $\{g_n^{(s,t)}\}_{n \geq 0}$ and $\{h_n^{(s,t)}\}_{n \geq 0}$, the following identity holds:*

$$g_{n+m}^{(s,t)} h_{n+l}^{(s,t)} - g_n^{(s,t)} h_{n+m+l}^{(s,t)} = (-1)^n (st)^n \left(g_m^{(s,t)} h_l^{(s,t)} - g_0^{(s,t)} h_{m+l}^{(s,t)} \right). \quad (2.28)$$

Proof. For fixed s, t , let $I_n^{(s,t)}$ defined by

$$I_n^{(s,t)} = g_{n+m}^{(s,t)} h_{n+l}^{(s,t)} - g_n^{(s,t)} h_{n+m+l}^{(s,t)}.$$

By Equation (2.20), we have

$$\begin{aligned} I_n^{(s,t)} &= g_{n+m}^{(s,t)} h_{n+l}^{(s,t)} - g_n^{(s,t)} h_{n+m+l}^{(s,t)} \\ &= [(st)J_{m-1}^{(s,t)} g_n^{(s,t)} + J_m^{(s,t)} g_{n+1}^{(s,t)}] h_{n+l}^{(s,t)} - g_n^{(s,t)} [(st)J_{m-1}^{(s,t)} h_{n+l}^{(s,t)} + J_m^{(s,t)} h_{n+l+1}^{(s,t)}] \\ &= (st)J_{m-1}^{(s,t)} g_n^{(s,t)} h_{n+l}^{(s,t)} + J_m^{(s,t)} g_{n+1}^{(s,t)} h_{n+l}^{(s,t)} - (st)g_n^{(s,t)} J_{m-1}^{(s,t)} h_{n+l}^{(s,t)} - J_m^{(s,t)} g_n^{(s,t)} h_{n+l+1}^{(s,t)} \\ &= J_m^{(s,t)} g_{n+1}^{(s,t)} h_{n+l}^{(s,t)} - J_m^{(s,t)} g_n^{(s,t)} h_{n+l+1}^{(s,t)} \\ &= J_m^{(s,t)} \left[g_{n+1}^{(s,t)} h_{n+l}^{(s,t)} - g_n^{(s,t)} h_{n+l+1}^{(s,t)} \right]. \end{aligned}$$

Analogously,

$$I_{n+1}^{(s,t)} = (st)J_m^{(s,t)} [g_n^{(s,t)} h_{n+l+1}^{(s,t)} - g_{n+1}^{(s,t)} h_{n+l}^{(s,t)}].$$

Then $I_{n+1}^{(s,t)} = -(st)I_n^{(s,t)}$. Hence,

$$I_1^{(s,t)} = -(st)I_0^{(s,t)}, \quad I_2^{(s,t)} = (-st)^2 I_0^{(s,t)}, \quad \dots, \quad I_n^{(s,t)} = (-st)^n I_0^{(s,t)}.$$

Thus, we arrive at Equation (2.28). $\square$

As a consequence of Theorem 2.10, we can specify for (s, t) -Jacobsthal sequence and (s, t) -Jacobsthal–Lucas sequence, the next result:

Corollary 2.11. For any integer $n, m, l \geq 0$, and for any positive real numbers s, t , the following properties are true:

$$j_{n+m}^{(s,t)} J_{n+l}^{(s,t)} - j_n^{(s,t)} J_{n+m+l}^{(s,t)} = (-1)^{n+1} (st)^n J_m^{(s,t)} j_l^{(s,t)},$$

$$J_{n+m}^{(s,t)} j_{n+l}^{(s,t)} - J_n^{(s,t)} j_{n+m+l}^{(s,t)} = (-1)^{n+1} (st)^n j_m^{(s,t)} J_l^{(s,t)},$$

where $\{J_n^{(s,t)}\}_{n \geq 0}$ denotes the (s, t) -Jacobsthal sequence, and $\{j_n^{(s,t)}\}_{n \geq 0}$ denotes the (s, t) -Jacobsthal–Lucas sequence.

2.3 Some classical identities

Using the Theorem 2.10 we find the Tagiuri-Vajda identity for the (s, t) -Jacobsthal sequence and the (s, t) -Jacobsthal–Lucas sequence.

Theorem 2.12 (Tagiuri–Vajda’s identity). *For any integers $n, m, l \geq 0$, and for any positive real numbers s, t , the following properties are true:*

$$J_{n+m}^{(s,t)} J_{n+l}^{(s,t)} - J_n^{(s,t)} J_{n+m+l}^{(s,t)} = (-1)^n (st)^n J_m^{(s,t)} J_l^{(s,t)}, \quad (2.29)$$

$$j_{n+m}^{(s,t)} j_{n+l}^{(s,t)} - j_n^{(s,t)} j_{n+m+l}^{(s,t)} = (-1)^{n+1} (st)^n (s+t)^2 J_m^{(s,t)} J_l^{(s,t)}, \quad (2.30)$$

where $J_n^{(s,t)}$ and $j_n^{(s,t)}$ represent the n -th terms of the (s, t) -Jacobsthal and (s, t) -Jacobsthal–Lucas sequences, respectively.

Proof. The Equation (2.28) allows us to make $g_n^{(s,t)} = h_n^{(s,t)} = J_n^{(s,t)}$, and we obtain

$$J_{n+m}^{(s,t)} J_{n+l}^{(s,t)} - J_n^{(s,t)} J_{n+m+l}^{(s,t)} = (-1)^n (st)^n (J_m^{(s,t)} J_l^{(s,t)} - J_0^{(s,t)} J_{m+l}^{(s,t)}),$$

and since $J_0^{(s,t)} = 0$, we obtain (2.29).

In a similar way, take $g_n^{(s,t)} = h_n^{(s,t)} = j_n^{(s,t)}$ in (2.28). By using (2.26), we obtain

$$\begin{aligned} j_{n+m}^{(s,t)} j_{n+l}^{(s,t)} - j_n^{(s,t)} j_{n+m+l}^{(s,t)} &= (-1)^n (st)^n (j_m^{(s,t)} j_l^{(s,t)} - j_0^{(s,t)} j_{m+l}^{(s,t)}) \\ &= (-1)^n (st)^n (j_m^{(s,t)} j_l^{(s,t)} - 2j_{m+l}^{(s,t)}) \\ &= (-1)^{n+1} (st)^n (s+t)^2 J_m^{(s,t)} J_l^{(s,t)}, \end{aligned}$$

which finishes the proof of (2.30). $\square$

Now, as a consequence of the Tagiuri-Vajda identity we obtain other important properties. In particular, Catalan’s identity, d’Ocagne’s identity and Cassini’s identity are also derived below.

Proposition 2.13 (d’Ocagne’s identity). *For any integers $h \geq 0$, $n \geq 0$ with $h \geq n$, and for any positive real numbers s, t , the following identities are valid:*

$$J_h^{(s,t)} J_{n+1}^{(s,t)} - J_n^{(s,t)} J_{h+1}^{(s,t)} = (-1)^{n+1} (st)^n J_{h-n}^{(s,t)} \quad (2.31)$$

$$j_h^{(s,t)} j_{n+1}^{(s,t)} - j_n^{(s,t)} j_{h+1}^{(s,t)} = (-1)^{n+1} (st)^n (s+t)^2 j_{h-n}^{(s,t)}, \quad (2.32)$$

where $J_n^{(s,t)}$ and $j_n^{(s,t)}$ represent the n -th terms of the (s, t) -Jacobsthal and (s, t) -Jacobsthal–Lucas sequences, respectively.

Proof. Consider $m + n = h$ and $l = 1$ in Equation (2.29) and we obtain the Equation (2.31). Similarly, make $m + n = h$ and $l = 1$ in Equation (2.30) and we obtain the Equation (2.32). $\square$

Proposition 2.14 (Catalan’s identity). *For any integers $m \geq 0$, $n \geq 0$ with $n \geq m$, and for any positive real numbers s, t , the following identities hold:*

$$(J_n^{(s,t)})^2 - J_{n-m}^{(s,t)} \cdot J_{n+m}^{(s,t)} = (-1)^{n+m+1} (st)^{n-m} (J_m^{(s,t)})^2, \quad (2.33)$$

$$(j_n^{(s,t)})^2 - j_{n-m}^{(s,t)} \cdot j_{n+m}^{(s,t)} = (-1)^{n+m} (st)^{n-m} (s+t)^2 (J_m^{(s,t)})^2, \quad (2.34)$$

where $J_n^{(s,t)}$ and $j_n^{(s,t)}$ represent the n -th terms of the (s, t) -Jacobsthal and (s, t) -Jacobsthal–Lucas sequences, respectively.

Proof. Taking $l = -m$ in Equation (2.29), and using (1.7) we get the Equation (2.33). Similarly, to obtain (2.34) take $l = -m$ in Equation (2.30). $\square$

Follows directly from Catalan’s identity the next identities.

Corollary 2.15. For any integer $n \geq 2$, and for any positive real numbers s, t , the following identities are valid true:

$$(J_n^{(s,t)})^2 - J_{n-2}^{(s,t)} \cdot J_{n+2}^{(s,t)} = (-1)^{n+1} (st)^{n-2} (s-t)^2, \quad (2.35)$$

$$(j_n^{(s,t)})^2 - j_{n-2}^{(s,t)} \cdot j_{n+2}^{(s,t)} = (-1)^n (st)^{n-2} (s+t)^2 (s-t)^2, \quad (2.36)$$

where $J_n^{(s,t)}$ and $j_n^{(s,t)}$ represent the n -th terms of the (s, t) -Jacobsthal and (s, t) -Jacobsthal–Lucas sequences, respectively.

Proposition 2.16 (Cassini-Simson’s identity). *For any integer $n \geq 1$ and for any positive real numbers s, t , the following identities hold:*

$$(J_n^{(s,t)})^2 - J_{n-1}^{(s,t)} \cdot J_{n+1}^{(s,t)} = (-1)^n (st)^{n-1} \quad (2.37)$$

$$(j_n^{(s,t)})^2 - j_{n-1}^{(s,t)} \cdot j_{n+1}^{(s,t)} = (-1)^{n-1} (st)^{n-1} (s+t)^2, \quad (2.38)$$

where $J_n^{(s,t)}$ and $j_n^{(s,t)}$ represent the n -th terms of the (s, t) -Jacobsthal and (s, t) -Jacobsthal–Lucas sequences, respectively.

Proof. It follows by doing $m = 1$ in Equations (2.33) and (2.34), and we get, respectively (2.37) and (2.38). $\square$

A direct consequence is the Cassini-Simson identity for even subscripts that is given above.

Corollary 2.17. For any integer $n \geq 1$ and for any positive real numbers s, t , the following identities are valid:

$$(J_{2n}^{(s,t)})^2 - J_{2n-1}^{(s,t)} \cdot J_{2n+1}^{(s,t)} = (st)^{2n-1} \quad (2.39)$$

$$(j_{2n}^{(s,t)})^2 - j_{2n-1}^{(s,t)} \cdot j_{2n+1}^{(s,t)} = -(st)^{2n-1}(s+t)^2, \quad (2.40)$$

where $J_n^{(s,t)}$ and $j_n^{(s,t)}$ denote the n -th term of the (s, t) -Jacobsthal and (s, t) -Jacobsthal-Lucas sequences, respectively.

The proofs using another approach to the identities of d'Ocagene, Catalan and Cassini-Simson can be found in [13].

To conclude the session, we present the identity of Gelin-Cesàro.

Proposition 2.18 (Gelin-Cesàro's identity). *For any integer $n \geq 2$ and for any positive real numbers s, t ; the following identities are true:*

$$\begin{aligned} J_{n+2}^{(s,t)} J_{n+1}^{(s,t)} J_{n-1}^{(s,t)} J_{n-2}^{(s,t)} - (J_n^{(s,t)})^4 &= (-1)^n (st)^{n-2} [st - (s-t)^2] (J_n^{(s,t)})^2 \\ &\quad - (s-t)^2 (st)^{2n-3}, \\ j_{n+2}^{(s,t)} j_{n+1}^{(s,t)} j_{n-1}^{(s,t)} j_{n-2}^{(s,t)} - (j_n^{(s,t)})^4 &= (j_n^{(s,t)})^2 (-1)^n (s+t)^2 (st)^{n-2} (st - (s-t)^2) \\ &\quad - (s-t)^2 (st)^{2n-3} (s+t)^4, \end{aligned}$$

where $J_n^{(s,t)}$ and $j_n^{(s,t)}$ denote the n -th term of the (s, t) -Jacobsthal and (s, t) -Jacobsthal-Lucas sequences, respectively.

Proof. Combining the Equations (2.35) and (2.37), we obtain

$$\begin{aligned} &J_{n+2}^{(s,t)} J_{n+1}^{(s,t)} J_{n-1}^{(s,t)} J_{n-2}^{(s,t)} - (J_n^{(s,t)})^4 \\ &= [(s-t)^2 + (-1)^{n+2} (st)^{n-2}] [(J_n^{(s,t)})^2 + (-1)^{n+1} (st)^{n-1}] - (J_n^{(s,t)})^4 \\ &= (J_n^{(s,t)})^2 (-1)^n (st)^{n-2} (st - (s-t)^2) - (s-t)^2 (st)^{2n-3}, \end{aligned}$$

as required.

Analogously, combining the Equations (2.36) and (2.38), and after some calculations, we get the last result. $\square$

3 One-parameter Jacobsthal-type sequence

In this section, we study a one-parameter generalization of Jacobsthal sequences; we give some new and well-known identities. Furthermore, we analyze the interrelationship between s -Jacobsthal and s -Jacobsthal-Lucas sequences.

In general, the results presented here will be obtained by specifying $t = 1$ in the results of the previous section; because of this, all proofs will be omitted. Another approach can be found, for instance, in [1, 2, 11, 15, 19].

3.1 Properties for the s -Jacobsthal numbers

As a consequence of the Definition 1.1, we obtain a homogeneous recurrence relation for each sequence.

Lemma 3.1 ([11], Equation (2)). *The s -Jacobsthal $\{J_n^{(s)}\}_{n \geq 0}$, and the s -Jacobsthal-Lucas $\{j_n^{(s)}\}_{n \geq 0}$ sequences satisfy the following properties:*

$$J_{n+1}^{(s)} = (s-1)J_n^{(s)} + sJ_{n-1}^{(s)}, \text{ for all } n \geq 1,$$

$$j_{n+1}^{(s)} = (s-1)j_n^{(s)} + sj_{n-1}^{(s)}, \text{ for all } n \geq 1,$$

where $J_0^{(s)} = 0$, $J_1^{(s)} = 1$, $j_0^{(s)} = 2$, and $j_1^{(s)} = s-1$.

Lemma 3.2. *For any integer $n \geq 0$ and any real number $s > 0$, the following identities hold:*

$$J_{n+1}^{(s)} = sJ_n^{(s)} + (-1)^n, \quad (3.1)$$

$$j_{n+1}^{(s)} = sj_n^{(s)} - (-1)^n(s+1), \quad (3.2)$$

where $\{J_n^{(s)}\}_{n \geq 0}$ is the s -Jacobsthal sequence, and $\{j_n^{(s)}\}_{n \geq 0}$ is the s -Jacobsthal-Lucas sequence.

The Equation (3.1) can be consulted in Theorem 1 from [1], and the Equation (3.2) in Theorem 2.1 from [15]. Furthermore, it can be obtained by making $t = 1$ in Proposition 2.2.

The next result is similar to Theorem 2.3 of Section 2.

Proposition 3.3. *For any integer $n \geq 0$ and any real number $s > 0$, the following identities are valid:*

$$2(s-1) \sum_{i=0}^n J_i^{(s)} = 2J_{n+1}^{(s)} + (-1)^{n+1} - 1, \quad (3.3)$$

$$2(s-1) \sum_{i=0}^n j_i^{(s,t)} = 2(j_{n+1}^{(s,t)} - 2) - (s+1)((-1)^{n+1} - 1), \quad (3.4)$$

where $\{J_n^{(s)}\}_{n \geq 0}$ is the s -Jacobsthal sequence, and $\{j_n^{(s)}\}_{n \geq 0}$ is the s -Jacobsthal–Lucas sequence.

Proof. Just make $t = 1$ in Equation (2.5) and obtain (3.3). Now, to get (3.4) replacing $t = 1$ in Equation (2.6). $\square$

It should be noted that, in the particular case when $s = 2$ in Equation (3.3), the known summation is obtained for the Jacobsthal numbers, in accordance with Exercise 71 from [8], that is,

$$2 \sum_{i=0}^n J_i = (2J_{n+1} + (-1)^{n+1}) - 1 = J_{n+2} - 1,$$

where $\{J_n\}_{n \geq 0}$ is the classical Jacobsthal sequence.

The s -Jacobsthal $\{J_n^{(s)}\}_{n \geq 0}$ also can be extended to negative subscripts. First we consider

$$\begin{aligned} J_{-n}^{(s)} &= \frac{s^{-n} - (-1)^{-n}}{s+1} = \frac{1}{(-1)^n s^n} \left(\frac{(-1)^n - s^n}{s+1} \right) \\ &= \frac{1}{(-1)^{n+1} s^n} \left(\frac{s^n - (-1)^n}{s+1} \right) = \frac{1}{(-1)^{n+1} s^n} J_n^{(s)}, \end{aligned}$$

for all $n \geq 0$. Under the previous discussion, the next definition is given:

$$J_{-n}^{(s)} = \frac{(-1)^{n+1}}{s^n} J_n^{(s)}, \text{ for all } n \geq 0. \quad (3.5)$$

3.2 Identities involving s -Jacobsthal-type sequence

In this subsection, the results allow us to express combinations between s -Jacobsthal sequence and s -Jacobsthal–Lucas sequence.

Proposition 3.4. *For any integer $n \geq 0$ and any real number $s > 0$, the following identities are true:*

$$\begin{aligned} (i) \quad & J_{n+2}^{(s)} + s^2 J_{n-2}^{(s)} = J_n^{(s)} j_2^{(s)}; \\ (ii) \quad & J_{n+2}^{(s)} - s^2 J_{n-2}^{(s)} = j_1^{(s)} j_n^{(s)}; \end{aligned}$$

$$(iii) \ J_{2n}^{(s)} = J_n^{(s)} j_n^{(s)};$$

where $\{J_n^{(s)}\}_{n \geq 0}$ is the s -Jacobsthal sequence, and $\{j_n^{(s)}\}_{n \geq 0}$ is the s -Jacobsthal–Lucas sequence.

Proof.

(i) It suffices to observe that

$$\begin{aligned} J_{n+2}^{(s)} + s^2 J_{n-2}^{(s)} &= \frac{s^{n+2} - (-1)^{n+2}}{s+1} + s^2 \frac{s^{n-2} - (-1)^{n-2}}{s+1} \\ &= \frac{s^{n+2} - (-1)^n + s^n - s^2(-1)^n}{s+1} \\ &= \frac{s^n(s^2 + 1) - (-1)^n(1 + s^2)}{s+1} \\ &= (s^2 + 1) \left(\frac{s^n - (-1)^n}{s+1} \right) = j_2^{(s)} J_n^{(s)}. \end{aligned}$$

(ii) We directly verify that

$$\begin{aligned} J_{n+2}^{(s)} - s^2 J_{n-2}^{(s)} &= \frac{s^{n+2} - (-1)^{n+2}}{s+1} - s^2 \frac{s^{n-2} - (-1)^{n-2}}{s+1} \\ &= \frac{s^2 s^n - (-1)^n - s^n + s^2(-1)^n}{s+1} \\ &= \frac{s^n(s^2 - 1) + (-1)^n(s^2 - 1)}{s+1} \\ &= (s^2 - 1) \frac{s^n + (-1)^n}{s+1} \\ &= (s - 1)[s^n + (-1)^n] = j_1^{(s)} j_n^{(s)}. \end{aligned}$$

(iii) It immediately follows that

$$\begin{aligned} J_{2n}^{(s)} &= \frac{s^{2n} - (-1)^{2n}}{s+1} = \frac{(s^n - (-1)^n)(s^n + (-1)^n)}{s+1} \\ &= \left(\frac{s^n - (-1)^n}{s+1} \right) (s^n + (-1)^n) = J_n^{(s)} j_n^{(s)}, \end{aligned}$$

as required. □

Notice that in Koshy [8] there are some identities involving the Jacobsthal sequence and Jacobsthal–Lucas sequence, for instance, that:

$$\begin{aligned} j_{n+m} &= J_{m+1}j_n + 2J_mj_{n-1}, \\ J_{n+m} &= J_{m+1}J_n + 2J_mJ_{n-1}, \\ 2J_{n+m} &= J_mj_n + 2J_nj_m, \end{aligned}$$

for any integers $m, n \geq 0$.

In the following result, we can see the corresponding generalization model for the s -Jacobsthal-type sequence. The verification is a similar procedure to that outlined in Proposition 3.4. For brevity, the proof will be omitted.

Proposition 3.5. *For any integer $n \geq 0$ and any real number $s > 0$, the following identities hold true*

$$\begin{aligned} (i) \quad j_{n+m}^{(s)} &= J_{m+1}^{(s)}j_n^{(s)} + sJ_m^{(s)}j_{n-1}^{(s)}; \\ (ii) \quad J_{n+m}^{(s)} &= J_{m+1}^{(s)}J_n^{(s)} + sJ_m^{(s)}J_{n-1}^{(s)}; \\ (iii) \quad 2J_{n+m}^{(s)} &= J_m^{(s)}j_n^{(s)} + J_n^{(s)}j_m^{(s)}. \end{aligned}$$

In a similar way to Propositions 3.4 and 3.5, we have:

Proposition 3.6. *For any integer $n \geq 1$ and any non negative real number $s > 1$, the following identities are valid:*

$$J_{n+1}^{(s)} + sJ_{n-1}^{(s)} = j_n^{(s)}; \quad (3.6)$$

$$J_{n+1}^{(s)} - sJ_{n-1}^{(s)} = j_1^{(s)}J_n^{(s)}; \quad (3.7)$$

$$(s-1)J_n^{(s)} + j_n^{(s)} = 2J_{n+1}^{(s)}; \quad (3.8)$$

$$j_{n+1}^{(s)} + sj_{n-1}^{(s)} = (s^2 - 1)J_n^{(s)}; \quad (3.9)$$

$$j_{n+1}^{(s)} - sj_{n-1}^{(s)} = j_n^{(s)}J_2^{(s)}, \quad (3.10)$$

where $\{J_n^{(s)}\}_{n \geq 0}$ is the s -Jacobsthal sequence, and $\{j_n^{(s)}\}_{n \geq 0}$ is the s -Jacobsthal–Lucas sequence.

Proof. We will prove the identities (3.9) and (3.10). The others are obtained in a similar

way. Note that

$$\begin{aligned}
 j_{n+1}^{(s)} + sj_{n-1}^{(s)} &= s^{n+1} + (-1)^{n+1} + s^n + s(-1)^{n-1} \\
 &= s^{n+1} + (-1)^{n+1} + s^n + s(-1)^{n+1} \\
 &= s^n(s+1) - (-1)^n(s+1) \\
 &= (s+1)(s-1) \frac{s^n - (-1)^n}{s-1} \\
 &= (s^2 - 1)J_n^{(s)},
 \end{aligned}$$

that is, we derive Equation (3.9). Now,

$$\begin{aligned}
 j_{n+1}^{(s)} - sj_{n-1}^{(s)} &= s^{n+1} + (-1)^{n+1} - s^n - s(-1)^{n-1} \\
 &= s^n(s-1) - (-1)^n + s(-1)^n \\
 &= (s-1)(s^n + (-1)^n) \\
 &= (s-1)j_n^{(s)},
 \end{aligned}$$

since $J_2^{(s)} = s-1$, we derive Equation 3.10. $\square$

Proposition 3.7. *For any integer $n \geq 1$ and any non negative real number $s > 1$, the following identities are valid:*

$$(s+1)^2 J_n^{(s)} J_{n+1}^{(s)} = j_{2n+1}^{(s)} - (-1)^n J_2^{(s)} (s)^n \quad (3.11)$$

$$(s+1)^2 (J_n^{(s)})^2 = j_{2n}^{(s)} - 2(-1)^n (s)^n \quad (3.12)$$

$$J_{n+1}^{(s)} j_n^{(s)} = J_{2n+1}^{(s)} + (-1)^n (s)^n \quad (3.13)$$

$$J_n^{(s)} j_{n+1}^{(s)} = J_{2n+1}^{(s)} - (-1)^n (s)^n \quad (3.14)$$

$$j_n^{(s)} j_{n+1}^{(s)} = j_{2n+1}^{(s)} + (-1)^n J_2^{(s)} (s)^n, \quad (3.15)$$

where $\{J_n^{(s)}\}_{n \geq 0}$ is the s -Jacobsthal sequence, and $\{j_n^{(s)}\}_{n \geq 0}$ is the s -Jacobsthal–Lucas sequence.

Proof. Just make $t = 1$ in Proposition 2.5, the desired results will be obtained. $\square$

An arbitrary generalized s -Jacobsthal sequence, or a s -Jacobsthal-type sequence, is defined by the s -Jacobsthal recurrence denoted as $\{g_n^{(s)}\}_{n \geq 0}$, that is, a $g_{n+1}^{(s)} = (s-1)g_n^{(s)} + sg_{n-1}^{(s)}$, for $n \geq 1$, with initial terms $g_0^{(s)} = a$ and $g_1^{(s)} = b$, where a and b are fixed real constants. The next Theorem is a specific case for the Theorem 2.6.

Theorem 3.8. For any integers $m, n \geq 0$, and for any real number $s > 1$, the generalized s -Jacobsthal sequence $\{g_n^{(s)}\}_{n \geq 0}$ satisfies the following identity

$$g_{n+m}^{(s)} = sJ_{m-1}^{(s)} \cdot g_n^{(s)} + J_m^{(s)} \cdot g_{n+1}^{(s)}, \quad (3.16)$$

where $\{J_n^{(s)}\}_{n \geq 0}$ is the s -Jacobsthal sequence.

Proof. By setting $t = 1$ in Equation (2.20), the desired result is obtained. $\square$

The next three results are consequences of the previous theorem.

Corollary 3.9. For any integers $m, n \geq 0$, and for any real number $s > 1$, the generalized s -Jacobsthal sequence $\{g_n^{(s)}\}_{n \geq 0}$ satisfies the following identity

$$(-1)^m (s)^m g_{n-m}^{(s)} = J_{m+1}^{(s)} \cdot g_n^{(s)} - J_m^{(s)} \cdot g_{n+1}^{(s)},$$

where $\{J_n^{(s)}\}_{n \geq 0}$ is the s -Jacobsthal sequence.

Corollary 3.10. For any integers n, m such that $m + n \geq 0$ and for any real number $s > 1$, the following identities hold:

$$\begin{aligned} J_{n+m}^{(s)} &= sJ_{m-1}^{(s)} \cdot J_n^{(s)} + J_m^{(s)} \cdot J_{n+1}^{(s)} \\ j_{n+m}^{(s)} &= sJ_{m-1}^{(s)} \cdot j_n^{(s)} + J_m^{(s)} \cdot j_{n+1}^{(s)}, \end{aligned}$$

where $\{J_n^{(s)}\}_{n \geq 0}$ is the s -Jacobsthal sequence and $\{J_n^{(s)}\}_{n \geq 0}$ is the s -Jacobsthal-Lucas sequence.

Corollary 3.11. For any integers n, m such that $n + m \geq 0$, $n - m \geq 0$ and for any real number $s > 1$, the following identities are valid:

$$\begin{aligned} 2J_{n+m}^{(s)} &= J_m^{(s)} \cdot j_n^{(s)} + J_n^{(s)} \cdot j_m^{(s)}, \\ 2(-1)^m (s)^m J_{n-m}^{(s)} &= J_n^{(s)} \cdot j_m^{(s)} - J_m^{(s)} \cdot j_n^{(s)}, \\ 2j_{n+m}^{(s)} &= (s+1)^2 J_n^{(s)} \cdot J_m^{(s)} + j_n^{(s)} \cdot j_m^{(s)}, \\ 2(-1)^m (s)^m j_{n-m}^{(s)} &= j_m^{(s)} \cdot j_n^{(s)} - (s+1)^2 J_m^{(s)} \cdot J_n^{(s)}, \end{aligned} \quad (3.17)$$

where $\{J_n^{(s)}\}_{n \geq 0}$ and $\{J_n^{(s)}\}_{n \geq 0}$ are, respectively, the s -Jacobsthal and the s -Jacobsthal-Lucas sequences.

We consider $\{g_n^{(s)}\}_{n \geq 0}$ and $\{h_n^{(s)}\}_{n \geq 0}$ two s -Jacobsthal-type sequences; in what follows, we will show some connections between them, and the next result is a particular case for $t = 1$ of Theorem 2.10.

Corollary 3.12. For any generalized s -Jacobsthal sequences $\{g_n^{(s)}\}_{n \geq 0}$ and $\{h_n^{(s)}\}_{n \geq 0}$, the following identity holds:

$$g_{n+m}^{(s)} h_{n+l}^{(s)} - g_n^{(s)} h_{n+m+l}^{(s)} = (-1)^n (s)^n (g_m^{(s)} h_l^{(s)} - g_0^{(s)} h_{m+l}^{(s)}). \quad (3.18)$$

In the next two results, we will apply Corollary 3.12 to specific cases of the s -Jacobsthal and the s -Jacobsthal-Lucas sequences.

Corollary 3.13. For any non negative integers $n, m, l \geq 0$, and for any real number $s > 1$, the following property is true:

$$j_{n+m}^{(s)} J_{n+l}^{(s)} - j_n^{(s)} J_{n+m+l}^{(s)} = (-1)^{n+1} s^n J_m^{(s)} j_l^{(s)},$$

where $\{J_n^{(s)}\}_{n \geq 0}$ and $\{j_n^{(s)}\}_{n \geq 0}$ are, respectively, the s -Jacobsthal and the s -Jacobsthal-Lucas sequences.

Proof. Take $g_n^{(s)} = j_n^{(s)}$ and $h_n^{(s)} = J_n^{(s)}$ in (3.18). Thus, by using (3.17) and some calculations, we have

$$\begin{aligned} j_{n+m}^{(s)} J_{n+l}^{(s)} - j_n^{(s)} J_{n+m+l}^{(s)} &= (-1)^n (s)^n (j_m^{(s)} J_l^{(s)} - j_0^{(s)} J_{m+l}^{(s)}) \\ &= (-1)^n (s)^n (j_m^{(s)} J_l^{(s)} - 2J_{m+l}^{(s)}) \\ &= (-1)^{n+1} s^n J_m^{(s)} j_l^{(s)}, \end{aligned}$$

as required. $\square$

Similar to Corollary 3.13 we have the next result.

Corollary 3.14. For any non negative integers $n, m, l \geq 0$, and for any real number $s > 1$, the following properties are true:

$$J_{n+m}^{(s)} j_{n+l}^{(s)} - J_n^{(s)} j_{n+m+l}^{(s)} = (-1)^{n+1} (s)^n j_m^{(s)} J_l^{(s)},$$

where $\{J_n^{(s)}\}_{n \geq 0}$ and $\{j_n^{(s)}\}_{n \geq 0}$ are, respectively, the s -Jacobsthal and the s -Jacobsthal-Lucas sequences.

3.3 Some classical identities

This subsection is dedicated to establish some classical identities for the s -Jacobsthal-type sequence.

We start finding the Tagiuri-Vajda identity for the $\{J_n^{(s)}\}_{n \geq 0}$ and $\{j_n^{(s)}\}_{n \geq 0}$ sequences, and many other interesting properties are derived.

Theorem 3.15 (Tagiuri–Vajda’s identity). *For any positive integers n, m, l , and for any real number $s > 1$, the following properties are true:*

$$J_{n+m}^{(s)} J_{n+l}^{(s)} - J_n^{(s)} J_{n+m+l}^{(s)} = (-1)^n (s)^n J_m^{(s)} J_l^{(s)}, \quad (3.19)$$

$$j_{n+m}^{(s)} j_{n+l}^{(s)} - j_n^{(s)} j_{n+m+l}^{(s)} = (-1)^{n+1} (s)^n (s+1)^2 J_m^{(s)} J_l^{(s)}, \quad (3.20)$$

where $\{J_n^{(s)}\}_{n \geq 0}$ is the s -Jacobsthal sequence and $\{j_n^{(s)}\}_{n \geq 0}$ is the s -Jacobsthal-Lucas sequence.

Proof. Just take $t = 1$ in Equation (2.29) and obtain (3.19). Now, taking $t = 1$ in Equation (2.30), Equation (3.20) is obtained. $\square$

These results were obtained using the same procedures as in Subsection 2.3, but this time with the equations from Theorem 3.15.

Proposition 3.16 (d’Ocagne’s identity). *For any integers $h \geq 0, n \geq 0$ with $h \geq n$, and for any real number $s > 1$, the following identities hold:*

$$\begin{aligned} J_h^{(s)} J_{n+1}^{(s)} - J_n^{(s)} J_{h+1}^{(s)} &= (-1)^{n+1} (s)^n J_{h-n}^{(s)} \\ j_h^{(s)} j_{n+1}^{(s)} - j_n^{(s)} j_{h+1}^{(s)} &= (-1)^{n+1} (s)^n (s+1)^2 j_{h-n}^{(s)}, \end{aligned}$$

where $\{J_n^{(s)}\}_{n \geq 0}$ denotes the s -Jacobsthal sequence, and $\{j_n^{(s)}\}_{n \geq 0}$ denotes the corresponding s -Jacobsthal–Lucas sequence.

Proposition 3.17 (Catalan’s identity). *For any integers $m \geq 0, n \geq 0$ with $n \geq m$, and for any real number $s > 1$, the following identities are true:*

$$\begin{aligned} (J_n^{(s)})^2 - J_{n-m}^{(s)} \cdot J_{n+m}^{(s)} &= (-1)^{n+m+1} (s)^{n-m} (J_m^{(s)})^2 \\ (j_n^{(s,t)})^2 - j_{n-m}^{(s)} \cdot j_{n+m}^{(s)} &= (-1)^{n+m} (s)^{n-m} (s+1)^2 (J_m^{(s)})^2, \end{aligned}$$

where $\{J_n^{(s)}\}_{n \geq 0}$ denotes the s -Jacobsthal sequence, and $\{j_n^{(s)}\}_{n \geq 0}$ denotes the corresponding s -Jacobsthal–Lucas sequence.

All these results follow directly from Catalan’s identity.

Corollary 3.18. For any integers $m \geq 0, n \geq 0$ with $n \geq m$, and for any positive real numbers s, t , the following identities are true:

$$\begin{aligned} (J_n^{(s)})^2 - J_{n-2}^{(s)} \cdot J_{n+2}^{(s)} &= (-1)^{n+1} (s)^{n-2} (s-1)^2, \\ (j_n^{(s)})^2 - j_{n-2}^{(s)} \cdot j_{n+2}^{(s)} &= (-1)^n (s)^{n-2} (s+1)^2 (s-1)^2, \end{aligned}$$

where $J_n^{(s)}$ and $j_n^{(s)}$ represent the n -th term of the (s) -Jacobsthal and (s) -Jacobsthal–Lucas sequences.

Proposition 3.19 (Cassini-Simson's identity). *For any integer $n \geq 1$ and for any non negative real numbers s , the following identities are valid true:*

$$\begin{aligned}(J_n^{(s)})^2 - J_{n-1}^{(s)} \cdot J_{n+1}^{(s)} &= (-1)^n (s)^{n-1}, \\ (j_n^{(s)})^2 - j_{n-1}^{(s)} \cdot j_{n+1}^{(s)} &= (-1)^{n+1} (s)^{n-1} (s+1)^2,\end{aligned}$$

where $\{J_n^{(s)}\}_{n \geq 0}$ and $\{j_n^{(s)}\}_{n \geq 0}$ are the s -Jacobsthal and s -Jacobsthal–Lucas sequences, respectively.

The Cassini-Simson identity for even subscripts is given below.

Corollary 3.20. For any integer $n \geq 1$ and for any real number $s > 1$, the following identities hold true:

$$\begin{aligned}(J_{2n}^{(s)})^2 - J_{2n-1}^{(s)} \cdot J_{2n+1}^{(s)} &= (s)^{2n-1}, \\ (j_{2n}^{(s)})^2 - j_{2n-1}^{(s)} \cdot j_{2n+1}^{(s)} &= -(s)^{2n-1} (s+1)^2,\end{aligned}$$

where $\{J_n^{(s)}\}_{n \geq 0}$ and $\{j_n^{(s)}\}_{n \geq 0}$ are the s -Jacobsthal and s -Jacobsthal–Lucas sequences, respectively.

We emphasize that the identities of the Catalan and Cassini-Simson for s -Jacobsthal-type sequence can also be deduced from Theorem 11 and Theorem 15, respectively, in [11].

The next result also follows directly from Catalan's identity.

Proposition 3.21 (Gelin-Cesàro's identity). *For any integer $n \geq 2$ and for any real number $s > 1$, the following identities are valid true:*

$$\begin{aligned}J_{n+2}^{(s)} J_{n+1}^{(s)} J_{n-1}^{(s)} J_{n-2}^{(s)} - (J_n^{(s)})^4 &= (J_n^{(s)})^2 (-1)^n (s)^{n-2} [s - (s-1)^2] \\ &\quad - (s-1)^2 (s)^{2n-3}, \\ j_{n+2}^{(s)} j_{n+1}^{(s)} j_{n-1}^{(s)} j_{n-2}^{(s)} - (j_n^{(s)})^4 &= (j_n^{(s)})^2 (-1)^n (s+1)^2 (s)^{n-2} (s - (s-1)^2) \\ &\quad - (s-1)^2 (s)^{2n-3} (s+1)^4,\end{aligned}$$

where $\{J_n^{(s)}\}_{n \geq 0}$ and $\{j_n^{(s)}\}_{n \geq 0}$ are the s -Jacobsthal and s -Jacobsthal–Lucas sequences, respectively.

4 Final Considerations

In this study, we have explored the (s, t) -Jacobsthal-type sequence, presenting it as a significant two-parameter extension of the classical Jacobsthal sequence. By establishing key identities, including extended versions of the Tagiuri-Vajda, d'Ocagne, Catalan, Cassini–Simson, and Gelin-Cesàro identities, we have expanded the understanding of this sequence and its algebraic properties. A crucial aspect of this research was the application of a novel approach, which yielded results that substantially complement existing findings in the literature. These new formulations strengthen the theoretical foundation of recurrence relations and open paths for future investigations into generalized Fibonacci-type sequences. Given the promising nature of these results, potential extensions of this work may involve exploring deeper connections with other special number sequences, as well as investigating their implications in combinatorial and number-theoretic contexts. We hope that these contributions serve as a useful reference for researchers working in related areas and inspire further advancements in the field.

These authors contributed equally to this work.

Acknowledgments

The third author is a member of the Research Centre CMAT-UTAD (Polo of Research Centre CMAT—Centre of Mathematics of University of Minho) and she thanks the Portuguese Funds through FCT—Fundação para a Ciência e a Tecnologia, within the Project UID/00013: Mathematics Centre of the University of Minho (CMAT/UM). The last author expresses his sincere gratitude to the Federal University of Tocantins (Arraias—Brazil) for its valuable support.

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Received 17 May 2025

Accept 25 September 2025