

LOTKA-VOLTERRA DYNAMICS UNDER IMPULSIVE EFFECTS AT VARIABLE TIMES

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Abstract

This paper investigates a Lotka-Volterra-type model subject to impulsive effects occurring at variable times. The motivation stems from the need to represent abrupt environmental changes, such as biological control interventions or resource management actions, which naturally introduce discontinuities in ecological systems. By modeling these disturbances through impulsive differential equations (IDEs), we explore the impact of state-dependent impulses on population dynamics. Particular emphasis is given to the analysis of periodic and non-periodic behaviors resulting from different impulsive strategies. The development of this work is supported by numerical simulations that illustrate the effectiveness of impulsive interventions and the conditions under which periodic solutions emerge.

Keywords: Impulsive differential equations, Lotka-Volterra model, periodic solutions.

1 Introduction

The study of differential equations remains an active area of research, both as a tool for modeling real-world problems and for purely mathematical analysis, due to their strong connection with the description of concrete phenomena. In several fields, such as

the physical, biological, and social sciences, the modeling of dynamical systems can be carried out using ordinary differential equations (ODEs), which establish relationships between variables and parameters, enabling the prediction of the system's behavior [2].

However, in classical ordinary differential equations, the prediction of the system's behavior depends solely on its current state, and its evolution occurs continuously. Therefore, this modeling framework is not always adequate for describing all natural phenomena [1], especially those that undergo abrupt changes over short time intervals, which are considered instantaneous. In such cases, one can employ impulsive differential equations (IDEs), which combine a differential equation with a condition that determines the occurrence of impulses, in addition to the initial condition.

Although Lakshmikantham (1969) provided the foundational theory for IDEs, in this work we focus on systems exhibiting discontinuous auto-oscillatory behavior, which are particularly relevant for ecological and biological contexts. Therefore, we adopt the definition and framework presented by Myshkis (1997), intrinsically linked to the theory of discontinuous dynamical systems and best suited for analyzing state-dependent impulses, which we call impulses in variable times. Additional results in this direction can be found in the papers by Rozko (1972) and Kaul (1990) [8, 5].

A discontinuous dynamical system in $\mathbb{R}^n$ can be defined by the following IDE, which combines continuous flow with discrete jumps:

$$\dot{x}(t) = f(x(t)), \quad x(t) \notin M, \quad (1.1)$$

$$x(t^+) = F(x(t)), \quad x(t) \in M. \quad (1.2)$$

Here, $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ is the vector field, $M \subset \mathbb{R}^n$ is a closed set (the impulsive set), and $F \in C(M, \mathbb{R}^n)$ is the impulsive function. The solution $x = \varphi(t, a)$ of the IDE (1.1)-(1.2) for a given initial condition

$$x(0) = a \quad (1.3)$$

is constructed as follows:

If $a \notin M$, then $\varphi(t, a)$ coincides with the solution to the continuous problem $\dot{x} = f(x)$ on the interval $[0, t_1]$, where $t_1 = t_1(a)$ is the first time $t > 0$ for which $\varphi(t, a) \in M$. (If no such t exists, $\varphi(t, a)$ is the solution of (1.1) for all $t > 0$ on its interval of existence).

At time t_1 , the state is reset to $\varphi(t_1^+, a) = F(\varphi(t_1, a))$ and there is $\epsilon > 0$, such that $\varphi(t, a) \notin M$, for all $t \in (t_1, t_1 + \epsilon)$. Then, for $t \in (t_1, t_2]$, $\varphi(t, a)$ is taken as the solution of equation (1.1) with the new initial condition $x(t_1) = \varphi(t_1^+, a)$, where t_2 is the next time $t > t_1$ for which $\varphi(t, a) \in M$.

If the initial state $a \in M$, we set $t_1 = 0$ and the described procedure is initiated immediately. This constructive process is formalized in the following definition by Myshkis (1997):

Definition 1.1. A function $\varphi(\cdot, a) : [0, T_a) \rightarrow \mathbb{R}^n$ ($0 < T_a \leq \infty$) is called a solution of (1.1)-(1.3), when the following conditions hold:

- i. It is continuous from the left, with $\varphi(0, a) = a$;
- ii. It is continuously differentiable and it satisfies (1.1) on some open interval with the left end t for any $t \in [0, T_a)$;
- iii. $\forall t \in [0, T_a) \Rightarrow$ if $\varphi(t^+, a) \notin M$, then $\varphi(t^+, a) = \varphi(t, a)$, but if $\varphi(t, a) \in M$ then $\varphi(t^+, a) = F(\varphi(t, a))$, and there is $\epsilon > 0$, such that for all $s \in (t, t + \epsilon)$, $\varphi(s, a) \notin M$.

To illustrate Definition 1.1, consider the two-dimensional case $n = 2$ where the flow is defined by $f(x, y) = (-1, 0)$, i.e.:

$$\begin{cases} \dot{x} = -1, \\ \dot{y} = 0. \end{cases} \quad (1.4)$$

Example 1.2. Consider the system (1.4) and the impulsive set $M = \{(x, y) \in \mathbb{R}^2 : x = 0\}$ and the impulse function $F(0, y) = (1, y)$.

Let $\varphi(t, a)$ be the solution with initial condition $a = (a_1, a_2)$, where $a_1 > 0$, then $\varphi(t, a)$ is expressed by:

$$\varphi(t, a) = (a_1 - t, a_2).$$

The trajectory reaches the impulsive set M when $x(t) = 0$, that is, at $t_1 = a_1$. The impact point is $\varphi(t_1, a) = (0, a_2)$, and after the impulse,

$$\varphi(t_1^+, a) = F(0, a_2) = (1, a_2).$$

From this new state, the trajectory continues as $\varphi(t, a) = (1 - (t - t_1), a_2)$ until the next impact at $t_2 = t_1 + 1$. The process repeats periodically with period $T = 1$.

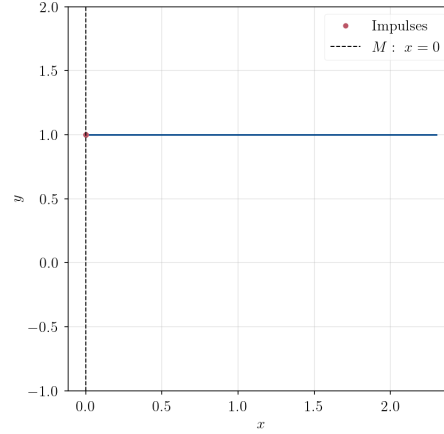
Hence, $\varphi(\cdot, a)$ has discontinuities at $t = a_1, a_1 + 1, a_1 + 2, \dots$ and is periodic for $t \geq a_1$, if $a_1 \geq 0$, as shown in Figure 1.

Example 1.3. Consider the system (1.4) and $M = \{(x, y) \in \mathbb{R}^2 : x = y\}$ and the impulsive function $F(x, y) = \left(x, \frac{y}{2}\right)$.

Let $\varphi(t, a)$ be the solution with initial condition $a = (a_1, a_2)$, where $a_1 \geq a_2 > 0$. The evolution is given by

$$\varphi(t, a) = (a_1 - t, a_2).$$

Figure 1: Orbit of $\varphi(\cdot, a)$ for Example 1.2.



Source: Created by the author.

The trajectory hits M when $x(t) = y(t)$, that is, at $t_1 = a_1 - a_2$. The impact point is $\varphi(t_1, a) = (a_2, a_2)$, and after the impulse,

$$\varphi(t_1^+, a) = F(a_2, a_2) = \left(a_2, \frac{a_2}{2}\right).$$

The next impact occurs at $t_2 = t_1 + \frac{a_2}{2}$, and the process repeats with

$$\varphi(t_k^+, a) = \left(\frac{a_2}{2^{k-1}}, \frac{a_2}{2^k}\right).$$

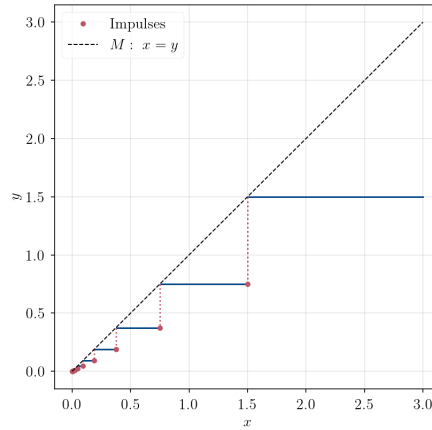
Each impact occurs at $\left(\frac{a_2}{2^{k-1}}, \frac{a_2}{2^{k-1}}\right)$, with inter-event times $\Delta t_k = \frac{a_2}{2^{k-1}}$. The sequence converges to the origin $(0, 0)$, and the total convergence time is

$$T = (a_1 - a_2) + \sum_{j=1}^{\infty} \frac{a_2}{2^j} = a_1.$$

Therefore, the solution $\varphi(\cdot, a)$ exhibits infinitely many impulses within a finite time interval, as illustrated in Figure 2, and it has discontinuity points at $t = a_1 - a_2, a_1 - \frac{a_2}{2}, a_1 - \frac{a_2}{2^2}, \dots$ and tends to 0^+ as $t \rightarrow a_1^-$, if $a_1 \geq a_2 > 0$.

Ecological systems, for instance, are subject to disturbances caused by human activities, such as resource exploitation and biological control, which can introduce discontinuities in the evolutionary process. These short-term disturbances are often modeled as impulses, representing sudden changes in the system's state.

Figure 2: Orbit of $\varphi(\cdot, a)$ for Example 1.3.



Source: Created by the author.

A typical case is the introduction of predators into an environment when the population of a given species reaches a critical threshold. This type of intervention can be characterized by impulses at variable times, which occur in the solution of the problem when it reaches a specific value, that is, when the population, during its development, reaches a predetermined threshold.

This phenomenon may affect both prey and predator populations. For example, if the prey population is too low, the predator population may be reduced to avoid the collapse of the food chain; on the other hand, if the prey population is too high, predators may be introduced to control their numbers and maintain ecological balance.

The analysis of the periodicity of impulses is particularly interesting, as populations tend to follow natural cycles, allowing for cost optimization in maintaining the balance of the studied system. This behavior is particularly relevant, given that removals typically follow regular intervals inherent to each species under study.

Consequently, IDEs have proven to be effective in representing natural phenomena, and through impulsive control, it is possible to establish criteria to prevent the extinction of all species by stabilizing some positive point, which may not necessarily correspond to the system's equilibrium point.

2 Lotka-Volterra model

One of the most well-known models for population dynamics is the Lotka–Volterra competition system, also known as the predator-prey model, independently proposed

by mathematicians Vito Volterra and Alfred James Lotka. In 1925, these equations were used to analyze predator-prey interactions, and the model has since become a classic, widely studied to this day. Due to its simplicity and biological relevance, it remains an essential tool in both theoretical and applied ecology.

Competition is an important biotic process that influences population dynamics in ecosystems. However, ecological systems are often disturbed by human exploitation and biological control activities, which may cause discontinuity in the process. Such models are subject to short-term disturbances, which are often assumed to be impulses in the modeling process. As a result, impulsive differential equations offer a more accurate representation of these systems and provide a natural framework for describing them [3, 4].

By simulating impulsive control, we obtain significant results where impulses are characterized as interventions (periodic or otherwise), such as pest releases, pesticide applications, or chemical poisoning in agricultural practices.

2.1 Case I: model without impulses

We first present the system without impulses:

$$\begin{cases} \dot{x} = x(r_1 - q_1 y), \\ \dot{y} = -y(r_2 - q_2 x), \\ x(0) = x_0, \\ y(0) = y_0, \end{cases} \quad (2.1)$$

where $x = x(t) > 0$ represents the prey biomass at time $t \geq 0$; $y = y(t) > 0$ denotes the predator biomass at time $t \geq 0$; the constants $r_1 > 0$ and $r_2 > 0$ are specific growth rate coefficients for prey and predator, respectively; $q_1 > 0$ and $q_2 > 0$ are intraspecific competition coefficients; and $x_0 > 0$, $y_0 > 0$ are the initial population levels of both species at $t = 0$.

The matrix form of the nonlinear system (2.1) is expressed as:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} r_1 & -q_1 x \\ q_2 y & -r_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

The equilibrium points of the system without impulses occur when there is no change in the prey or predator populations, which yields the critical points:

$$(0, 0) \text{ and } \left(\frac{r_2}{q_2}, \frac{r_1}{q_1} \right).$$

We first examine solutions near the origin $(0,0)$. By linearizing system (2.1), we obtain:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} r_1 & 0 \\ 0 & -r_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

We determine the eigenvalues from

$$\det \begin{pmatrix} \lambda - r_1 & 0 \\ 0 & \lambda + r_2 \end{pmatrix} = 0 \Rightarrow \lambda_1 = r_1 \text{ and } \lambda_2 = -r_2.$$

Since the eigenvalues have opposite signs, the critical point $(0,0)$ is an unstable point for system (2.1). This means that for positive initial condition near the origin, the trajectories diverge from it.

Next, we analyze the critical point $\left(\frac{r_2}{q_2}, \frac{r_1}{q_1}\right)$. To do so, we perform a change of variables to shift the equilibrium point to the origin, so $u = x - \frac{r_2}{q_2}$ and $v = y - \frac{r_1}{q_1}$.

Differentiating, we get:

$$\dot{u} = \dot{x} \quad \text{and} \quad \dot{v} = \dot{y}.$$

Hence,

$$\dot{u} = x(r_1 - q_1 y) = \left(u + \frac{r_2}{q_2}\right) \cdot \left[r_1 - q_1 \left(v + \frac{r_1}{q_1}\right)\right] = -q_1 \cdot \left(u + \frac{r_2}{q_2}\right) \cdot v,$$

$$\dot{v} = -y(r_2 - q_2 x) = -\left(v + \frac{r_1}{q_1}\right) \cdot \left[r_2 - q_2 \left(u + \frac{r_2}{q_2}\right)\right] = q_2 \cdot \left(v + \frac{r_1}{q_1}\right) \cdot u.$$

Hence, the matrix form of the system at this equilibrium point becomes:

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} -q_1 v & -q_1 \frac{r_2}{q_2} \\ q_2 \frac{r_1}{q_1} & q_2 u \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}.$$

After linearizing again using the variables u and v :

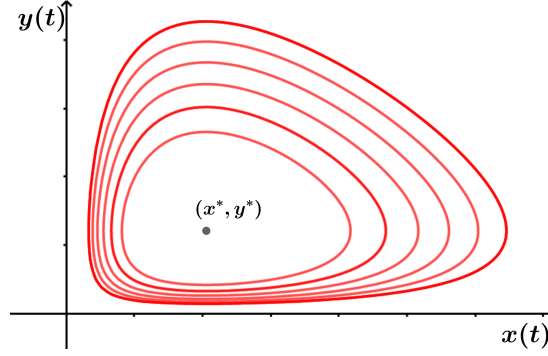
$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 & -q_1 \frac{r_2}{q_2} \\ q_2 \frac{r_1}{q_1} & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}.$$

The eigenvalues are determined by

$$\det \begin{pmatrix} \lambda & -q_1 \frac{r_2}{q_2} \\ q_2 \frac{r_1}{q_1} & \lambda \end{pmatrix} = 0 \Rightarrow \lambda = \pm i \sqrt{r_1 r_2}.$$

The eigenvalues are purely imaginary and thus the point $\left(\frac{r_2}{q_2}, \frac{r_1}{q_1}\right)$ is a center and is stable in the sense of Lyapunov. Thus, small perturbations cause closed orbits around this equilibrium, representing oscillations in predator and prey populations. To simplify notation, we denote this stable equilibrium by (x^*, y^*) .

Figure 3: Orbits of the predator-prey system without impulses.



Source: Created by the author.

The implicit equation describing these closed orbits is:

$$q_1 y - r_1 \ln y + q_2 x - r_2 \ln x = C, \quad (2.2)$$

where C is a positive constant, as illustrated in Figure 3.

Based on the analysis of the system without impulses, we now proceed with a clearer understanding of the model with impulses.

2.2 Case II: model with impulses at variable times

After analyzing the classical model, we introduce impulsive effects [3, 4]. Let $k > 0$ and define the impulsive set $M = \{(z, w) \in \mathbb{R}^2 | w = kz\}$. For $\alpha > 0$, we define the impulsive function $F : M \rightarrow \mathbb{R}^2$ by

$$F(z, w) = ((1 - \alpha)z, (1 - \alpha)w).$$

Thus, the impulsive prey-predator model is given by:

$$\begin{cases} \dot{x} = x(r_1 - q_1 y), \\ \dot{y} = -y(r_2 - q_2 x), \end{cases} \quad \text{if } (x(t), y(t)) \notin M.$$

Whenever the solution reaches the impulsive set M , that is, $(x(t), y(t)) \in M$, an instantaneous jump occurs according to

$$(x(t^+), y(t^+)) = F(x(t), y(t)).$$

The initial condition satisfies $(x(0), y(0)) = (x_0, y_0) \notin M$.

Equivalently, the impulsive prey-predator system can be written in the standard form of impulsive differential equations as

$$\begin{cases} \dot{x} = x(r_1 - q_1 y), & y(t) \neq kx(t), \\ \dot{y} = -y(r_2 - q_2 x), & y(t) \neq kx(t), \\ \Delta x(t) = -\alpha x(t), & y(t) = kx(t), \\ \Delta y(t) = -\alpha y(t), & y(t) = kx(t), \\ x(0) = x_0, \\ y(0) = y_0. \end{cases} \quad (2.3)$$

In system (2.3), the set of points (x, y) satisfying $y = kx(t)$, with $k > 0$ (to be defined later), determines the impulsive surface. The quantities $\alpha x(t)$ and $\alpha y(t)$ represent the amounts of prey and predator biomass, respectively, that are removed as impulses.

We define the right-hand side of equation (2.2) as $W(x, y)$, namely:

$$W(x, y) = q_1 y - r_1 \ln y + q_2 x - r_2 \ln x.$$

Let $U(x, y)$ be the difference between $W(x, y)$ evaluated at arbitrary points (x, y) and at the stable equilibrium (x^*, y^*) , this is the first integral of the system:

$$U(x, y) = W(x, y) - W(x^*, y^*).$$

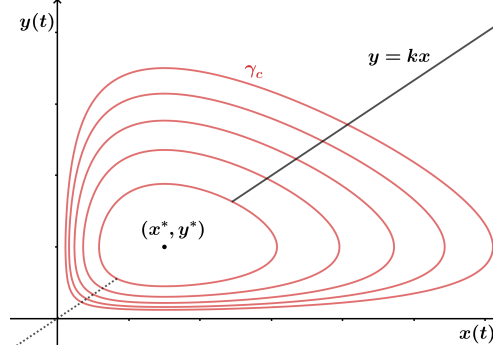
It follows that for all $(x, y) \in \mathbb{R}^+ \times \mathbb{R}^+$, $(x, y) \neq (x^*, y^*)$, the inequality $U(x, y) > 0$ holds, since $W(x, y) > 0$ for $x, y > 0$.

The orbits of system (2.3) are implicitly defined for some $c \geq 0$ as:

$$\gamma_c = \{(x, y) : U(x, y) = c\}. \quad (2.4)$$

and in Figure 4 we can see the orbits and the impulsive surface.

Figure 4: Orbits γ_c of system (2.2) and impulsive surface $y = kx$.



Source: Created by the author.

Assume the domain D , containing the orbit of the impulsive problem, lies between two boundary orbits γ_{c_1} and γ_{c_2} , with $0 < c_1 < c_2$, i.e., $D = D_{c_2} \setminus D_{c_1}$. Let c_0 be such that $c_1 < c_0 < c_2$. The orbit of the impulsive system corresponds to a segment of the orbit γ_{c_0} .

The impulsive surface (in this case, the segment where impulses occur) is a segment of the line $r : y = kx$, represented in Figure 4, and the slope of the line is chosen such that,

$$\frac{y^*}{a} \leq k \leq \frac{b}{x^*}, \quad (2.5)$$

where a and b are the largest solutions of the following equations:

$$U(a, y^*) = c_1 \Leftrightarrow q_2 a - r_2 \ln a = c_1 - r_2 (\ln(x^*) - 1),$$

$$U(x^*, b) = c_1 \Leftrightarrow q_1 b - r_1 \ln b = c_1 - r_1 (\ln(y^*) - 1).$$

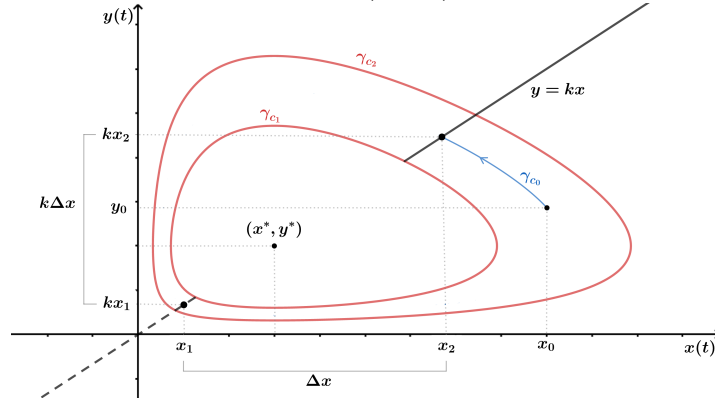
These equations determine the intersection points with the trajectories and from them, the estimate for the angular coefficient k is calculated.

The inequality (2.5) ensures that the impulsive line segment lies within the strip bounded by these two points. Hence, choosing k within this interval guarantees that the impulsive segment intersects the level sets γ_{c_1} and γ_{c_2} at their endpoints.

Therefore, it is possible to determine the impulsive set Φ so that the extremes (or the boundary) $\partial\Phi$ of the set belong to the trajectories γ_{c_1} and γ_{c_2} :

$$\Phi = \{(a, b); b - ka = 0, a > x^*, b > y^*\} \cap D_{c_2} \setminus D_{c_1}.$$

Figure 5: $U(x, kx) = c_0$.



Source: Created by the author.

From system (2.3), knowing that the impulsive surface is a segment of the line $y = kx$, the impulsive vector is expressed as:

$$B(x, y) = B(x, kx) = -(\alpha x, k\alpha x).$$

Thus, the solutions at impulsive times are given by:

$$(x^+, y^+) = (x^+, kx^+) = (x + \Delta x, kx + \Delta kx) = (x, kx) - \alpha(x, kx).$$

Assuming the initial condition of the impulsive problem (2.3) lies on the curve γ_{c_0} , at the moment when the orbit touches the impulsive surface, we have $U(x, kx) = c_0$, or equivalently,

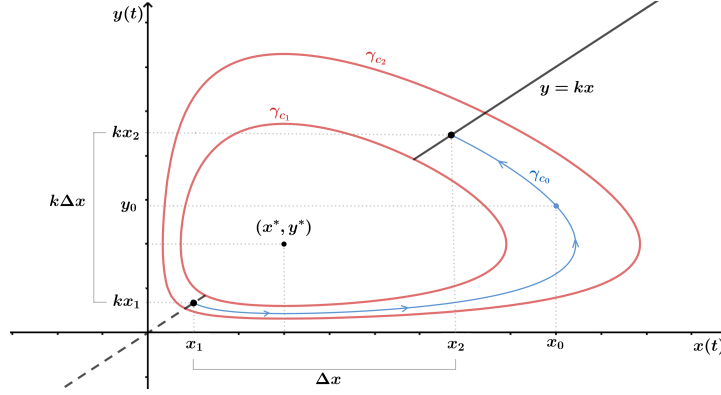
$$q_1 kx - r_1 \ln(kx) + q_2 x - r_2 \ln x = c_0 - r_1 \left(\ln \left(\frac{r_1}{q_1} \right) - 1 \right) - r_2 \left(\ln \left(\frac{r_2}{q_2} \right) - 1 \right). \quad (2.6)$$

Equation (2.6) admits only numerical solutions. Let these solutions be denoted by x_1 and x_2 with $x_1 < x_2$. Then $x = x_2$ and $y = kx_2$ are the approximate values of prey and predator populations at the impulse moment (removals).

In this case, we exhibit the periodic solution, in which the jump sizes are determined by the difference between the obtained solutions, such that:

$$\begin{cases} \Delta x(t) = x_2 - x_1, \\ \Delta kx(t) = k \cdot (x_2 - x_1). \end{cases}$$

Figure 6: Periodic orbit γ_{c_0} of the system.



Source: Created by the author.

The solution (x^+, y^+) after the impulse still lies on the orbit γ_{c_0} , constituting a new initial condition to continue the construction of the solution.

In constructing the solution to this impulsive problem, we observe that the impulse points and jump sizes coincide, giving rise to a periodic process. This is of particular interest, as removals tend to occur in specific periods for each species.

Since X_1 and X_2 are two points on the same closed orbit γ_{c_0} , with X_2 reached from X_1 following the continuous flow, let $T > 0$ denote the corresponding travel time, so that

$$\varphi(T, X_1) = X_2.$$

Moreover, the impulse is chosen such that

$$F(X_2) = X_1.$$

Hence, after the continuous evolution from X_1 to X_2 and the subsequent impulsive jump, the solution returns to X_1 , and the process repeats periodically with period T .

2.3 Numerical solution

In the specific case shown in Figure 6, phase space plots of $x \times y$ were generated for the system's orbits, γ_{c_i} . These orbits correspond to the constants $c_1 = 2$, $c_2 = 4$, and $c_0 = 3$ from equation (2.4), which satisfy the condition $c_1 < c_0 < c_2$.

The coefficients arbitrarily chosen for the model were $r_1 = 4, r_2 = 3, q_1 = 2, q_2 = 1$, and thus we obtain the equilibrium point $(x^*, y^*) = (3, 2)$ and solving the equation (2.5), we choose $k = \frac{2}{3}$, using the parameters in system (2.3):

$$\begin{cases} \dot{x} = x(4 - 2y), & y(t) \neq \frac{2}{3} x(t), \\ \dot{y} = -y(3 - x), & y(t) \neq \frac{2}{3} x(t), \\ \Delta x(t) = -0.85 x(t), & y(t) = \frac{2}{3} x(t), \\ \Delta y(t) = -0.85 y(t), & y(t) = \frac{2}{3} x(t), \\ x(0) = 9, \\ y(0) = 2.86. \end{cases} \quad (2.7)$$

The amount α of removal at the impulse moments in (2.7) is calculated in order to obtain a periodic solution, thus solving

$$U\left(x, \frac{2}{3}x\right) = 3,$$

the approximate values found for x were $x_1 = 1$ and $x_2 = 6.69$. The intersection points between the orbit and the impulsive surface are $X_1 = (1, 0.67)$ and $X_2 = (6.69, 4.46)$. Since $x_2 > x_1$, the point X_2 undergoes an impulse.

The magnitude of the jump, as represented in (2.7), is approximate, but can be obtained, since the jump is $\Delta x(t) = \alpha x(t)$, so the trajectory reaches the line at $X_2 = (6.69, 4.46)$ and the post-jump state is:

$$X_2^+ = X_2 + \Delta X_2 = X_2 - \alpha X_2 = X_2(1 - \alpha).$$

For the cycle to be periodic, the post-jump state X_2^+ must be equal to the starting point of the cycle, $X_1 = (1, 0.67)$. Therefore, the condition for periodicity is $X_2(1 - \alpha) = X_1$, which allows us to calculate α :

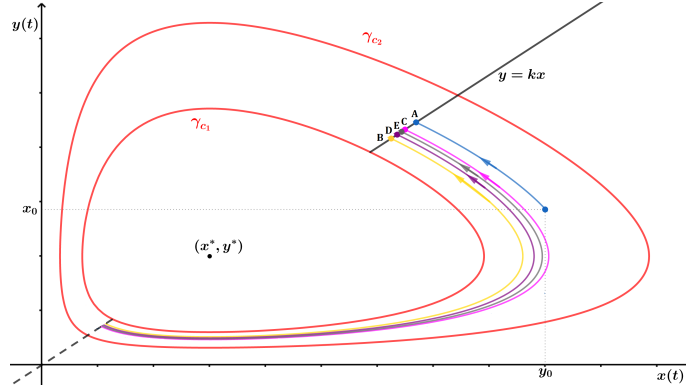
$$6.69(1 - \alpha) = 1 \Rightarrow (1 - \alpha) \approx 0.149 \Rightarrow \alpha \approx 0.85.$$

We obtain the same approximate value for α from $\Delta y(t)$, confirming that the jump occurs on the curve γ_{c_0} and, particularly, at the intersection point between the orbit and the impulsive surface, thus characterizing a periodic process.

In this example, the intersection points with the impulsive surface are $X_1 = (1, 0.67)$ and $X_2 = (6.69, 4.46)$. The trajectory starting at X_1 evolves along the orbit γ_{c_0} until it reaches X_2 , where the impulse occurs. Since the impulse maps X_2 back to X_1 , the resulting impulsive solution is periodic.

On the other hand, the removals could occur in a non-periodic manner. In this case, after the impulse, the new initial condition would belong to a different orbit.

Figure 7: Orbit of the non-periodic impulsive solution.



Source: Created by the author.

For that, using system (2.7) and setting the parameter $\alpha = 0.83$, we obtain a non-periodic behavior of the solution, as illustrated in Figure 7.

The initial condition is given on the orbit $U(x, kx) = 3$, with $(x_0, y_0) = (9, 2.86)$. From this initial point, the orbit is represented by the blue trajectory, and the first impulse point is $A = (6.69, 4.47)$.

The point $A^+ = A - \alpha A$. Since we set $\alpha = 0.83$, we have $A^+ = (1.13, 0.75)$, which becomes the new initial condition for a new orbit with $c = 2.44$, represented by the yellow curve.

The orbit intersects the impulsive surface at point $B = (6.24, 4.17)$, and after the impulse, $B^+ = (1.06, 0.70)$, generating a new orbit with $c = 2.74$, shown in pink.

Following this process, we determine the points $C = (6.49, 4.33)$, $D = (6.35, 4.24)$, $E = (6.43, 4.29)$ and their respective orbits, represented in Figure 7 in purple and gray, to illustrate the non-periodic behavior resulting from the initially chosen jump size.

3 Conclusion

Impulsive differential equations (IDEs) are essential tools for modeling systems that undergo abrupt changes due to punctual interventions, such as sudden removals, inputs, or environmental shifts. These discontinuities, far from being a limitation, allow the model to better capture real-world behaviors that are not continuous in nature but still follow consistent and analyzable patterns.

The predator-prey model with impulses, for instance, enables the representation of external actions, such as population removals or reinforcements, at specific moments in time, providing a framework to study coexistence while avoiding extinction. The

periodic case allows the analysis of the stable dynamics of the system with removals, allowing behavior to be predicted with greater confidence. On the other hand, in the non-periodic case, the irregularity of the impulses generates more varied orbits, making it difficult to accurately estimate population growth or decline.

Moreover, by applying numerical methods, one can address inverse problems, such as estimating the amount of population to be removed given a desired period of intervention, which further reinforces the practical relevance of impulsive models in biological and environmental management.

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