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CENTRAL LIMIT THEOREM FOR DIMENSION OF MEASURE IN CIRCLE

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Abstract

We prove a version of the Central Limit Theorem for the dimension of the equilibrium state of expanding endomorphisms defined on $\mathbb{S}^1$. The results here can be extended to conformal expanding maps defined on n-dimensional differentiable manifolds.

Keywords: Dimension of measure, Central Limit Theorem, Expanding maps, Equilibrium states, Gibbs measure.

1 Introduction

We will consider expanding maps defined on the circle $\mathbb{S}^1$. The focus is to study the pointwise dimension d_μ of the measure μ (see the definition below). This quantity is related to the Lyapunov exponent λ and the Kolmogorov entropy h_μ. We will study the limit defined in Equation 2.1. More precisely, we will study the convergence

$$\frac{\log \mu(B(x,\varepsilon)) - d_\mu \log \varepsilon}{\sqrt{-\log \varepsilon}},$$

and prove that this limit converges in distribution to the Wiener process. Using the same ideas, we can extend to conformal expanding maps defined on differentiable manifolds of dimension n. The non-conformal case was addressed by Leplaideur and Saussol in

[8] for a skew-product on the torus $\mathbb{T}^n$, using the semiconjugacy to the shift map to apply symbolic dynamics results, such as the preservation of the Gibbs property of a measure via a factor map. For more information on the results of preserving the Gibbs property of a measure, see [3, 4, 5, 10]. The main relevance of this paper's results is that we address the conformal case, which was not studied in [8]. They limited themselves to a skew product map defined on the n-dimensional torus and used the existence of a Markov partition to semiconjugate it to the shift map. Here, our maps are more general and defined in an n-dimensional differentiable manifold, answering (for the conformal case) two questions asked in [8]. We do not use the structure of a partition nor symbolic dynamics results, thus differentiating the technique used here. In short, the idea is that, in the part of the paper where we use concepts from Dynamic Systems, we use topological properties to approximate the ball $B(x, \varepsilon)$ by dynamic balls $B_n(x, \varepsilon)$, using appropriate expansion times. Hence, we strongly rely on the property that the measure is Gibbs. From there, we use probabilistic arguments to study the fluctuation S_n and apply the Principle of Invariance.

2 Statement of the main result

It is well known that expanding maps have the uniqueness of the equilibrium states μ for a Holdër continuous potential and that it is a Gibbs measure. See [2, 11] for more details. Associated with this measure, we have the notion of pointwise dimension of μ.

$$d_\mu(x) = \lim_{\varepsilon \to 0} \frac{\log \mu(B(x, \varepsilon))}{\log \varepsilon}, \tag{2.1}$$

whenever the limit exists. It is known (see [7]) that for μ-almost every point x, the pointwise dimension $d_\mu(x) = d_\mu$ exists in the context of expanding map. We prove a version of the Central Limit Theorem for the pointwise dimension of this equilibrium state. We denote by $D([0, 1])$ the set of cadlag functions in $[0, 1]$ endowed with the Skorohod topology.

Theorem 2.1. *Let $f : \mathbb{S}^1 \to \mathbb{S}^1$ be an expanding endomorphism $C^{1+\alpha}$, $\phi : \mathbb{S}^1 \to \mathbb{R}$ be a Hölder continuous potential, and μ be the equilibrium state for (f, ϕ). Then, the process*

$$\frac{\log \mu(B(x, \varepsilon^t)) + t d_\mu \log \varepsilon}{\sqrt{-\log \varepsilon}}, \quad \textit{for every } \ t \in [0, 1] \tag{2.2}$$

converges in $D([0, 1])$ in distribution for the Wiener process $W(t)$.

3 Preliminaries

3.1 Concepts of Probability Theory

In this section, we will discuss some concepts from Probability Theory used in the text.

Definition 3.1. *A $\mathbb{R}^d$-value vector X is called Gaussian if it has a multidimensional charateristic function $\phi_X : \mathbb{R}^d \to \mathbb{C}$ given by*

$$\phi_X(s) = E[e^{is \cdot X}] = e^{-\langle s, Vs \rangle/2 + i\langle m, s\rangle},$$

for some nonsingular symmetric $n \times n$ matrix V and vector $E[X]$.

The matrix V is called *the covariance matrix*, and the vector m is called the *mean vector*.

Definition 3.2. *A continuous Gaussian process $X(t)$, $0 \le t \le 1$, with a mean vector $m_t = E[X(t)]$ and covariance matrix $V(s,t) = Cov[X(s), X(t)] = E[(X(s) - m_s) \cdot (X(t) - m_t)^*]$ is called the Wiener process (or Brownian motion) if for any $0 \le t_0 < t_1 < \cdots t_n$, the random vectors $X(0), X(t_{i+1}) - X(t_i)$ are independent and the covariance matrix V satisfies $V(s,t) = V(r,r)$ where $r = \min(s,t)$ and $s \to V(s,s)$.*

We say that $X(t)$ converges in distribution for X, written $X(t) \overset{\mathcal{D}}{\Longrightarrow} X$, if $E[g(X_n)] \to E[g(X)]$ for every g bounded, continuous and real function.

Theorem 3.1 (Slutsky's Theorem)**.** *Let $X(t)$ and $Y(t)$ be two sequences of random vectors such that $X(t) \overset{\mathcal{D}}{\Longrightarrow} X$ and $Y(t) \overset{P}{\Longrightarrow} c$, where c is a constant. Let g be a continuous function. Then*

$$g(X(t), Y(t)) \overset{\mathcal{D}}{\Longrightarrow} (X, c).$$

We will use the Skorohod topology because, *a priori*, the stochastic processes studied here are discontinuous. Furthermore, the space of cadlag functions endowed with the uniform convergence norm is non-separable, which can cause many problems. See [1] for more details. We denote by $D([0,1])$ the set of cadlag functions in $[0,1]$ endowed with the Skorohod topology.
We say that $g, h \in D([0,1])$ are α-close if there exists $\beta : [0,1] \to [0,1]$ such that

1. β is increasing and $\beta(0) = 0$ and $\beta(1) = 1$;
2. $|\beta(t) - t| \le \alpha$ for all $t \in [0,1]$;

3. $|g(\beta(t)) - h(t)| \leq \alpha$ for all $t \in [0, 1]$.

We consider random constants and/or random processes over the probability space $(\mathbb{S}^1, \mathcal{B}, \mu)$, where $\mathcal{B}$ is the Borel σ-field. Constants are constants with respect to the parameter ε.

Given two random variables, X_ε e Y_ε, we use the notation $X_\varepsilon \approx Y_\varepsilon$ to mean that there exists a constant $\tau > 0$ such that for μ-a.e. x, we have

$$\sup_{0<\varepsilon<\tau} |X_\varepsilon(x) - Y_\varepsilon(x)| < +\infty. \tag{3.1}$$

3.2 Concepts of Dynamical Systems Theory

In this section, we will discuss some concepts from Dynamical Systems Theory used in the text.

Definition 3.3. *A C^1 endomorphism $f : \mathbb{S}^1 \to \mathbb{S}^1$ is expanding if there exists $\sigma > 1$ such that $|Df(x)| > \sigma$ for all $x \in \mathbb{S}^1$.*

A differentiable map f is conformal if the derivative at each point is a nonzero scalar multiple of an orthogonal transformation, i.e., if the derivative contracts or expands distances by the same amount in all directions. Obviously, in dimension one, any nonsingular differentiable map is conformal.

A measure μ is f-invariant if $\mu(f^{-1}(B)) = \mu(B)$ for each $B \in \mathcal{B}$. And μ is an ergodic measure if $\mu(B) = 0$ or 1 for each set $B \in \mathcal{B}$ f-invariant. Let $\mathbb{P}_f(\mathbb{S}^1)$ be the set of f-invariant probability measures, and $\phi : \mathbb{S}^1 \to \mathbb{R}$ be continuous observable. The connection between topological pressure and the metric entropy of an invariant measure is formalized by the variational principle. We refer the reader to [12] for a proof of the Variational Principle:

$$P_f(\phi) = \sup_{\eta \in \mathbb{P}_f(\mathbb{S}^1)} \left\{ h_\eta(f) + \int \phi \, d\eta \right\}, \tag{3.2}$$

where $h_\eta(f)$ is the metric entropy of f.

The Variational Principle gives a natural way to choose interesting measures of $\mathbb{P}_f(\mathbb{S}^1)$. We say that $\mu \in \mathbb{P}_f(\mathbb{S}^1)$ is an *equilibrium state* for (f, ϕ) if μ satisfies

$$P_f(\phi) = h_\mu(f) + \int \phi \, d\mu.$$

We define the Lyapunov exponent $\lambda(x)$ in x by

$$\lambda(x) = \lim_{n\to\infty} \frac{1}{n} \log |Df^n(x)|,$$

if the limit exists. By Oseledet's Theorem, see [12], for a f-invariant measure μ, the Lyapunov exponent $\lambda(x)$, there exists μ-a.e. point $x \in \mathbb{S}^1$. If the measure μ is ergodic, $\lambda(x)$ is constant μ-a.e. point x.

Theorem 3.2 (Birkhoff Ergodic Theorem, [12]). *Let $f : \mathbb{S}^1 \to \mathbb{S}^1$ be a measurable transformation and μ be a probability measure f-invariant and ergodic. Given any integrable function $\phi : \mathbb{S}^1 \to \mathbb{R}$. Then, μ-almost every point $x \in \mathbb{S}^1$*

$$\lim_{n\to\infty} \frac{1}{n} S_n\phi(x) = \int \phi d\mu,$$

where $S_n\phi(x) := \sum_{j=0}^{n-1} \phi(f^j(x))$.

Given $\varepsilon > 0$, $n \in \mathbb{N}$ and $x \in X$, consider the *dynamical ball*

$$B_n(x, \varepsilon) = \left\{ y \in X : \ d(f^j(x), f^j(y)) < \varepsilon, \text{ for } \ 0 \le j < n \right\}.$$

We also say that μ is a *Gibbs measure* for (f, ϕ) if for some constants $P \in \mathbb{R}$ and $\delta > 0$ the following holds: for all $\varepsilon \le \delta$ there exists $K = K(\varepsilon) > 0$ so that for μ-almost every $x \in \mathbb{S}^1$ we have

$$K^{-1} \le \frac{\mu(B_n(x, \varepsilon))}{\exp(S_n\phi(y) - nP)} \le K, \tag{3.3}$$

for all $y \in B_n(x, \varepsilon)$ and all $k \ge 1$.

Remark 3.1. *The equation in (3.3) in the language of probability theory means $\mu(B_n(x, \varepsilon)) \approx e^{S_n\phi(x)}$. See equation (3.1) above.*

4 Proof Theorem 2.1

In this section, we prove Theorem 2.1. To do this, we will use the following standard results on expanding maps, which can be found in [11, 12].

Lemma 4.1 (Distortion lemma). *Let f be an $C^{1+\alpha}$ expanding endomorphism of $\mathbb{S}^1$. There is a constant $c > 0$ such that if $I \subset \mathbb{S}^1$ is an interval and f^n is injective on I then*

$$c^{-1} < \frac{|Df^n(y)|}{|Df^n(z)|} < c \quad \textit{for any} \ \ y, z \in I.$$

Proposition 4.1. *Let $f : \mathbb{S}^1 \to \mathbb{S}^1$ be an C^1 expanding endomorphism. There exists $\rho > 0$ such that, for any pre-image x of any point $y \in \mathbb{S}^1$, there exists a map f_y^{-1} : $B(y, \rho) \to \mathbb{S}^1$ of class C^1 such that $f^n \circ f_y^{-n} = id$ and $f_y^{-n}(y) = x$ and*

$$d(f_y^{-n}(w), f_y^{-n}(z)) \leq \sigma^{-n} d(f^n(w), f^n(z)),$$

for each $w, z \in B(y, \rho)$ and for $n \geq 1$.

The following lemma provides us with a relationship between the dynamic ball $B_n(x, \varepsilon)$ and the ball $B(x, \varepsilon)$, exploiting the property of the map f being conformal. Let ρ be the constant of Proposition 4.1.

Lemma 4.2. *In μ-a.e., there exists a noncrescent sequence $(\kappa_n)_{n\geq 1} = (\kappa_n(x))_{n\geq 1}$ such that for each $0 < \varepsilon < \rho$, we can choose $n_\varepsilon = n(\varepsilon) \geq 1$ satisfying $B(x, \kappa_{n_\varepsilon+1}\varepsilon) \subseteq B(x, \varepsilon) \subseteq B(x, \kappa_{n_\varepsilon}\varepsilon)$. Moreover, $B_{n_\varepsilon+1}(x, \varepsilon) \subseteq B(x, \varepsilon) \subseteq B_{n_\varepsilon}(x, \varepsilon)$.*

Proof. For each n and x we consider the inverse branch f_x^{-n} defined on ball $B(f^n(x), \rho)$ such that $f^n(x)$ is mapped on x.

Let $(\kappa_n(x))_i$ be the sequence formed by the contraction rate accumulated of x, i.e., $\kappa_n(x) = \prod_{j=1}^{n} \|Df^{-1}(f^j(x))\|$. Since the rate is uniform, we have $\kappa_n(x) = \kappa_n = \sigma^{-n}$.

For $0 < \varepsilon < \rho$ small enough, choose the longest time n_ε such that $B(x, \kappa_{n_\varepsilon+1}\varepsilon) \subseteq B(x, \varepsilon)$. Since f is conformal,

$$B_{n_\varepsilon+1}(x, \varepsilon) = B(x, \kappa_{n_\varepsilon+1}\varepsilon) \subseteq B(x, \varepsilon) \subseteq B(x, \kappa_{n_\varepsilon}\varepsilon) = B_{n_\varepsilon}(x, \varepsilon),$$

finishing the proof. □

Remark 4.1. *Note that by the Distortion Lemma 4.1, there exists a constant $C > 0$ such that $C \leq \kappa_{n_\varepsilon}\varepsilon \leq 1$. Obviously $\varepsilon \to 0$ iff $n_\varepsilon \to \infty$.*

Assuming the hypothesis of Theorem 2.1, let μ be the unique state of equilibrium for (f, ϕ). The Proposition 4.2 below is the ***key result*** in the proof of Theorem 2.1.

Proposition 4.2. *In μ-a.e. $x \in \mathbb{S}^1$ and for each $t \in [0, 1]$ if*

$$N''_\varepsilon(t) := \frac{S_{n_\varepsilon t}\phi(x)}{\sqrt{-\log \varepsilon}} \xrightarrow{\mathcal{D}} W(t)$$

then

$$N_\varepsilon(t) := \frac{\log \mu(B(x, \varepsilon^t)) - td\mu \log \varepsilon}{\sqrt{-\log \varepsilon}} \xrightarrow{\mathcal{D}} W(t),$$

where $W(t)$ is the Wiener process.

4.0.1 Proof of Proposition 4.2

For the proof of Proposition 4.2, the idea is to study the convergence of several intermediate processes. First, be the process N'_ε given by

$$N'_\varepsilon(t) = \frac{\log \mu(B(x, \kappa_{n_{\varepsilon^t}})) - td\mu \log \varepsilon}{\sqrt{-\log \varepsilon}}, \qquad t \in [0, 1].$$

Lemma 4.3. *If $N'_\varepsilon(t) \xrightarrow{\mathcal{D}} W(t)$ then $N_\varepsilon(t) \xrightarrow{\mathcal{D}} W(t)$.*

Proof. Suppose that $N'_\varepsilon(t)$ converges in distribution for a process $W(t)$. We define the following process

$$P_\varepsilon(t) = \frac{\log \mu(B(x, \kappa_{n_{\varepsilon^t}+1})) - td_\mu \log \varepsilon}{\sqrt{-\log \varepsilon}} \quad \text{and} \quad Q_\varepsilon(t) = \frac{\log \mu(B(x, \kappa_{n_{\varepsilon^t}})) - td_\mu \log \varepsilon}{\sqrt{-\log \varepsilon}}.$$

Note that $P_\varepsilon \leq N_\varepsilon \leq Q_\varepsilon$ and $P_\varepsilon(t), Q_\varepsilon(t)$ converge in distribution for the same process. We claim that the process $R_\varepsilon(t) = Q_\varepsilon(t) - P_\varepsilon(t)$ converges to zero in probability. In fact, in μ-a.e. and $t \in [0, 1]$ we have

$$\begin{aligned} |R_\varepsilon(t)| &= \left| \frac{\log \mu(B(x, \kappa_{n_{\varepsilon^t}+1})) - \log \mu(B(x, \kappa_{n_{\varepsilon^t}}))}{\sqrt{-\log \varepsilon}} \right| \\ &\leq \frac{1}{\sqrt{-\log \varepsilon}} \cdot C \end{aligned}$$

where C is a positive constant.

Then, for each $\tau > 0$, we have $\lim_{\varepsilon \to 0} \mathbb{P}(|R_\varepsilon(t)| \geq \tau) = 0$, that is, $R_\varepsilon(t) \xrightarrow{P} 0$. Therefore, by the Slutsky theorem, we have $N_\varepsilon(t) \xrightarrow{\mathcal{D}} W(t)$.

□

Lemma 4.4. *For $t \in [0, 1]$, in μ-a.e. x we have*

$$\lim_{\varepsilon \to 0} \frac{n_{\varepsilon^t}(x)}{-\log \varepsilon^t} = \frac{1}{\lambda},$$

where $\lambda = \log |Df|$ is the Lyapunov exponent.

Proof. By Remark 4.1,

$$\begin{aligned} C \leq \kappa_{n_\varepsilon} \varepsilon \leq 1 &\iff \frac{1}{n_\varepsilon} \log C \leq \log \frac{1}{n_\varepsilon} \kappa_{n_\varepsilon} + \frac{1}{n_\varepsilon} \log \varepsilon \leq \frac{1}{n_\varepsilon} \log 1 \\ &\iff \lim_{\varepsilon \to 0} \log \kappa_{n_\varepsilon} = \lim_{\varepsilon \to 0} -\log \varepsilon. \end{aligned}$$

By the Birkhoff's Ergodic Theorem

$$\lim_{\varepsilon\to 0}\frac{n_\varepsilon(x)}{-\log\varepsilon}=\lim_{\varepsilon\to\infty}\frac{n_\varepsilon(x)}{\log\kappa_{n_\varepsilon}}=\lim_{n\to\infty}\left(\frac{-\log(\prod_{j=1}^{n}\|Df^{-1}(f^j(x))\|)}{n}\right)^{-1}=\lambda^{-1}.$$

□

An interesting fact is that, with the arguments above, we can provide a proof of the formula for the pointwise dimension, establishing the relation between the Lyapunov exponent λ and the pointwise dimension d_μ.

Lemma 4.5. *With the notations before, we have*

$$d_\mu=\frac{\int\phi d\mu}{\lambda}.$$

Proof. In μ- a.e. x we have by Lemma 4.4 and Birkhoff's Ergodic Theorem

$$d_\mu=\lim_{\varepsilon\to 0}\frac{\log\mu(B(x,\varepsilon))}{-\log\varepsilon}\quad=\quad\lim_{\varepsilon_i\to 0}\frac{n_\varepsilon}{-\log\varepsilon}\frac{S_{n_\varepsilon}\phi(x)}{n_\varepsilon}=\frac{1}{\lambda}\int\phi d\mu.$$

□

Now, let us consider the process of Proposition 4.2, that is,

$$N''_\varepsilon(t):=\frac{S_{n_\varepsilon t}\phi}{\sqrt{-\log\varepsilon}}. \tag{4.1}$$

We have the following relationship between the two processes N' and N''.

Lemma 4.6. *There exists a finite constant K a.e. such that for all $t\in[0,1]$, we have:*

$$\sup_{t\in[0,1]}|N''_\varepsilon(t)-N'_\varepsilon(t)|\leq\frac{K}{\sqrt{-\log\varepsilon}}$$

for each ε small enough.

Proof. Using language in probability theory(see remark 3.1), we will prove the following statement: In $\mu-$a.e. x, we have

$$\log\mu(B(x,\kappa_{n_\varepsilon}\varepsilon))-d_\mu\log\varepsilon\approx S_{n_\varepsilon}\phi(x).$$

In fact, since $B(x,\kappa_{n_\varepsilon}\varepsilon)=B_{n_\varepsilon}(x,\varepsilon)$ and μ is a Gibbs measure($\mu(B_{n_\varepsilon}(x,\varepsilon))\approx e^{S_{n_\varepsilon}\phi(x)}$), we have $B(x,\kappa_{n_\varepsilon}\varepsilon)\approx e^{S_{n_\varepsilon}\phi(x)}$. On the other hand, we have

$$-d_\mu \log \varepsilon \approx d_\mu \log \prod_{j=1}^{n_\varepsilon} \|Df^{-1}(f^j(x))\| \approx \frac{\int \phi d\mu}{\lambda} \cdot \lambda n_\varepsilon = n_\varepsilon \int \phi d\mu \approx S_{n_\varepsilon}\phi.$$

Then, there exists a constant K, such that for $t \in [0,1]$ we have $|N''_\varepsilon(t) - N'_\varepsilon(t)| \leq \frac{K}{\sqrt{-\log \varepsilon}}$. □

Note that for each $\tau > 0$, we have $\lim_{\varepsilon \to 0} \mathbb{P}(|N''_\varepsilon(t) - N'_\varepsilon(t)| \geq \tau) = 0$. Therefore, by the Slutsky theorem, if $N''_\varepsilon(t) \overset{\mathcal{D}}{\Longrightarrow} W(t)$ then $N'_\varepsilon(t) \overset{\mathcal{D}}{\Longrightarrow} W(t)$. We now proceed with the completion of the demonstration of the main proposition.

Conclusion of the proof of Proposition 4.2.

Proof. For $\varepsilon > 0$ sufficiently small, by Lemma 4.3 we have process $N_\varepsilon(t) \approx N'_\varepsilon(t)$. And by Lemma 4.6, $N''_\varepsilon(t) \approx N'_\varepsilon(t)$. And this concludes the proof of the proposition. □

4.0.2 Invariance Principle

The Invariance Principle states that certain normalized random processes—such as partial sums of random variables—converge in distribution to a continuous process: Brownian motion. In other words, it approximates the trajectories of $S_n\phi$ by Brownian motion. Since the observable ϕ is Holder continuous, $S_n\phi$ satisfies the Central Limit Theorem. However, a direct application is not possible because the times n_ε depend on the point. From here, the proof closely follows Leplaideur and Saussol [8]. To show convergence of the process N''_ε, we consider the subsequence $\varepsilon = e^{-k}$, that is,

$$X_k = N''_{e^{-k}} = \frac{S_{n_{e^{-k}}}\phi}{\sqrt{k}}.$$

We define the process Y_k by

$$Y_k(t) = \frac{1}{\sqrt{k}} \sum_{j=0}^{\lfloor kt \rfloor - 1} \phi(f^j(x)) + \frac{1}{\sqrt{k}}(kt - \lfloor kt \rfloor)\phi(f^{\lfloor kt \rfloor}).$$

It is well known that the Weak Invariance Principle is stated as follows. This result can be found in [9].

Theorem 4.1. *The process Y_k converges in distribution on $C([0,1],\mathbb{R})$ for the Brownian motion $W = (W(t))_{t\in[0,1]}$.*

Remark 4.2. *If n_ε is independent of the process Y_k, the conclusion of Theorem 2.1 follows immediately.*

Conclusion of the proof of Theorem 2.1. We fix $a > \frac{1}{\lambda}$. We define the process $Z_k \in C([0,a],\mathbb{R})$ defined by $Z_k(t) = Y_k\mid_{[0,a]}$ and Φ_k on $C([0,1],[0,a])$ given by

$$\Phi_k(t) = \begin{cases} \frac{\nu_k(t)}{k}, & \text{se} \quad \frac{\nu_k(t)}{k} \leq a \\ \frac{1}{\lambda}, & \text{otherwise,} \end{cases}$$

where $\nu_k(t)$ are continuous functions $n_{e^{-kt}}$ obtained from a polynomial interpolation in discontinued points. Then

$$X_k(t) = Z_k \circ \Phi_k(t) - O(\frac{1}{\sqrt{k}}).$$

Following the arguments of Billingsley([1], p.152), we prove that $X_k(t) \overset{\mathcal{D}}{\Longrightarrow} W$. By Lemma 4.4,

$$\frac{n_{\varepsilon^t}(x,y)}{-\log \varepsilon^t} \xrightarrow{\varepsilon\to 0} \frac{t}{\lambda} \quad \text{then} \quad \frac{n_{\varepsilon^t}(x,y)}{-\log \varepsilon^t} \xrightarrow{P} \frac{t}{\lambda}.$$

As a consequence, the map Φ_k converges almost surely to the uniform convergence norm for the map $\Phi(t) = \frac{t}{\lambda}$. And finally, for the conclusion of the proof, we have the lemma.

Lemma 4.7. $X_k \overset{\mathcal{D}}{\Longrightarrow} X = W \circ \Phi.$

Proof. As $Z_k \overset{\mathcal{D}}{\Longrightarrow} W$ and $\Phi_k \xrightarrow{P} \Phi$ then $(Z_k, \Phi_k) \overset{\mathcal{D}}{\Longrightarrow} (,\Phi)$. By the continuity of the composition of the maps, $Z_k \circ \Phi_k$ converges for $B \circ \Phi$. Therefore, $X_k \overset{\mathcal{D}}{\Longrightarrow} X$. Clearly, X is a Wiener process. □

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References

[1] Billingsley, P.: Convergence of Probability Measures. In: *Wiley Series in Probability and Statistics: Probability and Statistic* 2nd (John Wiley & Sons,1999).

[2] Bowen, R. Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms. volume 470 of *Lect. Notes in Math.* Springer Verlag, 1975.

[3] Chazottes, J. ;Ugalde, E.: Projection of Markov Measures may be Gibbsian. *Journal of Statistical Physics* 111(5-6):1245–1272, 2003.

[4] Chazottes, J. ;Ugalde, E.: On the Preservation of Gibbsianness under Symbol Amalgamation. *In Brian Marcus, Karl Petersen, and Tsachy Weissman, editors, Entropy of Hidden Markov Processes and Connections to Dynamical Systems*, pages 72–97. Cambridge University Press, 2011. Cambridge Books Online.

[5] Ferreira, G.; Oliveira, K. : Sequential Gibbs Measures and Factor Maps, *J Stat Phys*, 172, 833–853 (2018).

[6] Kempton, T. : Factors of Gibbs Measures for Subshifts of Finite Type. *Bulletin of the London Mathematical Society*, 43(4):751–764, 2011.

[7] Ledrappier, F. ; Young, L.-S. : The Metric Entropy of Diffeomorphisms: Part I: Characterization of Measures Satisfying Pesin's Entropy Formula. *Annals of Mathematics* (1985), pp. 509–539.

[8] Leplaideur, R. ; Saussol, B. : Central Limit Theorem por Dimension of Gibbs Measures in Hyperbolic Dynamics. *Stochastics and Dynamics* 12 (2012), 24.

[9] Melbourne, Ian; Nicol, Matthew: A Vector-Valued Almost Sure Invariance Principle for Hyperbolic Dynamical Systems. *Ann. Probab.* 37 (2009) 478–505.

[10] Pollicott, M. and Kempton, T. : Factors of Gibbs Measures for Full Shifts. *In Brian Marcus, Karl Petersen, and Tsachy Weissman, editors, Entropy of Hidden Markov Processes and Connections to Dynamical Systems*, pages 246–257. Cambridge University Press, 2011. Cambridge Books Online.

[11] Ruelle, David : Thermodynamic Formalism. *Encyclopedia of Mathematics and its Applications 5*, Addison-Wesley Publishing Co., Reading, Mass (1978)

[12] Walters, P.: An Introduction to Ergodic Theory. *Graduate Texts in Mathematics.* Bd. 79:1 . Springer-Verlag New York, 2000

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