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CONGRUENCES OF NARAYANA NUMBERS MODULO PRIMES AND POWERS OF PRIMES

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Abstract

Sivaraman establishes in [13] a structural identity between Narayana central numbers and Catalan numbers. Based on his ideas and using classical results from the literature, as Fermat's Little Theorem, Lucas' Theorem and Wolstenholme's Theorem, this paper derives necessary and sufficient conditions for a Narayana central number to be divisible by an arbitrary prime.

Keywords: Narayana numbers, Narayana triangle, Congruences.

1 Introduction

Narayana numbers, denoted by $N(n,r)$, are a sequence of natural numbers defined by the formula $N(n,r) = \frac{1}{k}\binom{n}{k-1}\binom{n-1}{k-1}$, for $1 \leq k \leq n$. They appear in various areas of combinatorics, especially in counting discrete structures such as paths, trees, and triangulations. These numbers, which are the subject of this paper, were named in honor of Tadepalli Venkata Narayana (1930 - 1987), a Canadian mathematician of Indian origin. His notable contribution was to systematize and demonstrate the relevance of these numbers in lattice path theory (for more details, see [6]).

Although the elements of the Narayana triangle appeared implicitly in earlier works, it was T.V. Narayana's in-depth study, particularly in the 1950s, that consolidated their use and presented an explicit formula for counting paths from $(0,0)$ to (n,n) that do not cross the diagonal $x = y$. This work highlighted connections to Dyck paths with a fixed number of peaks. In turn, such paths have applications in various fields, such as the encoding of triangulations of convex polygons - which are associated with Catalan numbers - and the representation of rooted planar trees with a prescribed number of leaves.

It is worth noting that contemporary interest in this topic remains active, with several works proposed in the recent literature. A generalization is presented in the paper *On Generalized Narayana Numbers* [15]. Other generalizations such as [3], [7], and [16] discuss extensions and additional properties associated with the classical Narayana numbers.

In this paper we explore a specific family of Narayana numbers known as Narayana central numbers. We develop results on the divisibility of these numbers by arbitrary primes and their powers. To establish the main divisibility results, our methodological approach is grounded in classical number-theoretic theorems and known congruences. We employ tools such as Wolstenholme's Theorem, Fermat's Little Theorem, and Lucas' Theorem. Furthermore, to derive further congruences for the central Narayana numbers, we leverage known congruences for the Catalan numbers (as seen in the works of [5] and [8]) and extend them by exploiting the close relationship between these two sequences, as formalized by Theorem 3.2.

Thus, we organized the work into two sections: in Section 2, we recall classical definitions and results on Catalan and Narayana numbers. In Section 3, we develop new congruences for $N(2n,n)$ modulo any prime p and its powers. Finally, conclusions and future perspectives are presented.

2 Preliminary results

This section presents some well-known results from the literature to aid in the formulation and proof of subsequent ones. We will establish some properties of Catalan and Narayana numbers.

Catalan numbers are a family of natural numbers well-known in both combinatorics and number theory. Although they were defined by the mathematician Eugène Charles Catalan (1814-1894), they were already known to Leonard Euler and Johann Andreas von Segner in their work on decomposing convex polygons into triangles by means of their diagonals (see [4]). This family of numbers is formally defined as follows.

Definition 2.1. The Catalan Numbers C_n, for $n \geq 1$ and with initial condition $C_0 = 1$, are given by the recurrence

$$\begin{aligned} C_n &= C_{n-1} \cdot C_0 + C_{n-2} \cdot C_1 + C_{n-3} \cdot C_2 + \cdots + C_1 \cdot C_{n-2} + C_0 \cdot C_{n-1} \\ &= \sum_{k=0}^{n-1} C_{n-k-1} \cdot C_k \end{aligned} \tag{2.1}$$

It follows from identity (2.1) that

$$\begin{aligned} C_1 &= C_0 \cdot C_0 = C_0^2 = 1^2 = 1, \\ C_2 &= C_1 \cdot C_0 + C_0 \cdot C_1 = 1 \cdot 1 + 1 \cdot 1 = 1 + 1 = 2, \\ C_3 &= C_2 \cdot C_0 + C_1^2 + C_0 \cdot C_2 = 2 \cdot 1 + 1^2 + 1 \cdot 2 = 2 + 1 + 2 = 5, \\ C_4 &= C_3 \cdot C_0 + C_2 \cdot C_1 + C_1 \cdot C_2 + C_0 \cdot C_3 = 5 \cdot 1 + 2 \cdot 1 + 1 \cdot 2 + 1 \cdot 5 = 14. \end{aligned}$$

Definition 2.1 provides a recursive method for obtaining the Catalan numbers. Table 1 presents the first 30 Catalan numbers.

Table 1: The first 30 Catalan numbers

n	C_n	n	C_n
0	1	15	9694845
1	1	16	35357670
2	2	17	129644790
3	5	18	477638700
4	14	19	1767263190
5	42	20	6564120420
6	132	21	24466267020
7	429	22	91482563640
8	1430	23	343059613650
9	4862	24	1289904147324
10	16796	25	4861946401452
11	58786	26	18367353072152
12	208012	27	69533550916004
13	742900	28	263747951750360
14	2674440	29	1002242216651368

Petersen established in [11] the ordinary generating function for the sequence $\{C_n\}_{n\geq 0}$ of Catalan numbers. Furthermore, using the Generalized Binomial Theorem, he determined an explicit formula for the general term C_n, which is given by the next result.

Proposition 2.2. *([11], Theorem 2.1) For all $n \geq 0$ the explicit formula for the n-th Catalan number is given by*

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

We will now present a sequence of numbers that is the main focus of this paper and will be denoted by $N(n,k)$, for $1 \leq k \leq n$. We call this sequence the Narayana Numbers, and according to [11], they refine the Catalan numbers. Alternatively, we can think of Narayana numbers as the number of Dyck paths with exactly r peaks (for more details, see also [11]). The next definition follows from [10].

Definition 2.3. For all $n \geq 1$ e $1 \leq k \leq n$, the Narayana numbers are determined by the following formula

$$N(n,k) = \frac{1}{k}\binom{n}{k-1}\binom{n-1}{k-1}. \tag{2.2}$$

Note that (2.2) can be rewritten as

$$\begin{aligned} N(n,k) = \frac{1}{k}\binom{n}{k-1}\binom{n-1}{k-1} &= \frac{1}{k}\frac{n!}{(k-1)!(n-k+1)!}\frac{(n-1)!}{(n-k)!(k-1)!} \\ &= \frac{1}{n}\frac{n!}{k!(n-k)!}\frac{n!}{(n-k+1)!(k-1)!} \\ &= \frac{1}{n}\binom{n}{k}\binom{n}{k-1}. \end{aligned} \tag{2.3}$$

From (2.3), the following result holds, which provides another explicit formula for $N(n,k)$ and will be used throughout this work.

Proposition 2.4. *([16], Equation (1.1)) For all $n \geq 1$ and $1 \leq k \leq n$ we have*

$$N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}.$$

In Table 2 we present the values of $N(n,k)$, for $1 \leq n \leq 5$ and $1 \leq k \leq n$. Furthermore, for each row in this table, we sum the entries and obtain the corresponding Catalan numbers for each $n = 1, 2, \cdots, 5$. This is a known result and is presented in the next proposition.

Table 2: Narayana Numbers Triangle

$n\backslash k$	1	2	3	4	5	6	7	8	$\sum_{k=1}^{n} N(n,k) = C_n$
1	1	–	–	–	–	–	–	–	1
2	1	1	–	–	–	–	–	–	2
3	1	3	1	–	–	–	–	–	5
4	1	6	6	1	–	–	–	–	14
5	1	10	20	10	1	–	–	–	42
6	1	15	50	50	15	1	–	–	132
7	1	21	105	175	105	21	1	–	429
8	1	28	196	490	490	196	28	1	1430

Proposition 2.5. *[16] Let C_n be the n-th Catalan numbers. For all $n \geq 1$ and $1 \leq k \leq n$, we have $\sum_{k=1}^{n} N(n,k) = C_n$.*

Proposition 2.6 is presented in [14] and is proven using only the recurrence relation for Catalan numbers and the parity of n.

Proposition 2.6. *Let n be a non-negative integer. If n is even, then $C_n \equiv 0 \pmod 2$.*

Note that Proposition 2.6 tells us that if n is even, then C_n is even. The converse is not valid. A counterexample is $C_5 = 42$.

Following a line of reasoning analogous to the one adopted in Proposition 2.6, it is possible to draw conclusions about the value of C_n when n is a non-negative odd integer.

Proposition 2.7. *(see [14], Theorem 2) Given a non-negative integer n. If n is odd, then $C_n \equiv C^2_{\frac{n-1}{2}} \pmod 2$.*

The following corollary will be important for characterizing the parity of the Catalan numbers and, consequently, the central Narayana numbers (see Propositions 3.3 and 3.4). It introduces a necessary and sufficient condition for C_n to be odd, in addition to establishing a relationship between the indices n and $\frac{n-1}{2}$.

Corollary 2.8. Given a non-negative odd integer n, then

$$C_n \equiv \begin{cases} 0 \pmod 2, & \text{if, and only if, } C_{\frac{n-1}{2}} \text{ is even,} \\ 1 \pmod 2, & \text{if, and only if, } C_{\frac{n-1}{2}} \text{ is odd.} \end{cases}$$

Proof. Let us prove the case for odd n. Suppose that $C_n \equiv 1 \pmod 2$, so C_n is odd. Let $w = C_0 \cdot C_{n-1} + C_1 \cdot C_{n-2} + \cdots + C_{\frac{(n-3)}{2}} \cdot C_{\frac{(n+1)}{2}}$. Since n is odd, if follows that $C_n = 2w + C^2_{\frac{n-1}{2}}$. As C_n is odd by hypothesis and $2w$ is even, then, in the last equality, $C^2_{\frac{n-1}{2}}$ is odd, which implies $C_{\frac{n-1}{2}}$ is odd.

Conversely, suppose $C_{\frac{n-1}{2}}$ is odd. Since n is odd, we have again that $C_n = 2w + C^2_{\frac{n-1}{2}}$. As $C_{\frac{n-1}{2}}$ is odd and $2w$ is even, it follows that C_n is odd, that is, $C_n \equiv 1 \pmod 2$. ■

In the next theorem we present an interesting result that characterizes the parity of Catalan numbers and involves Mersenne numbers, that is, numbers of the form $2^k - 1$, for some natural number k. This result was established by Sivaraman in ([14], Theorem 2). To prove it, the author uses the recurrence relation for Catalan numbers (see Definition 2.1) and determines the values of n for which C_n is odd. When the process stops, we obtain that C_n is odd if, and only if, $n = 2^k - 1$, that is, when n is a Mersenne number.

Theorem 2.9. *A Catalan number C_n is odd if, and only if, n is a Mersenne number.*

In Table 3 we illustrate the result given by Theorem 2.9. Note that in the interval [7, 15] the only odd Catalan numbers occur when $n = 7 = 2^3 - 1$ and when $n = 15 = 2^4 - 1$. The remaining numbers are all even.

Table 3: Parity of C_n

n	C_n
7	429
8	1430
9	4862
10	16796
11	58786
12	208012
13	742900
14	267440
15	9694845

The following proposition is an important result for this article, which provides us with a congruence for binomial coefficients modulo a prime number p.

Proposition 2.10. *(Lucas' Theorem, see [9]) Let p be a prime number and n, k two non-negative integers such that $n = \sum_{j=0}^{s} n_j p^j$ and $k = \sum_{j=0}^{s} k_j p^j$. Then,*

$$\binom{n}{k} \equiv \prod_{j=0}^{s} \binom{n_i}{k_i} \pmod{p}.$$

Let's see an example.

Example 2.11. Consider $p = 3, n = 5$ and $k = 2$. Thus, $\binom{5}{2} = 10$ e $10 \equiv 1 \pmod 3$. On the other hand, we have that $5 = 2 \cdot 3^0 + 1 \cdot 3^1$ e $2 = 2 \cdot 3^0 + 0 \cdot 3^1$. Thus,

$$\prod_{j=0}^{1} \binom{n_i}{k_i} = \binom{1}{0}\binom{2}{2} = 1.$$

That is, $\binom{5}{2} \equiv \binom{1}{0}\binom{2}{2} \equiv 1 \pmod 3$.

The next result shows that, if p is prime, there is no base $a < p$ with $\gcd(a, p) = 1$ such that $a^{p-1} - 1$ has a non-zero residue modulo p. This result is known as Fermat's Little Theorem and is of great importance for the divisibility of integers (for more details see [17]).

Proposition 2.12. *Let p be a prime number and $a \in \mathbb{N}$ such that $\gcd(p, a) = 1$. Then, $a^p \equiv a \pmod p$ or $a^{p-1} \equiv 1 \pmod p$.*

Note that the absence of a non-zero residue in Fermat's Little Theorem does not guarantee that p is prime. For example, let $n = 341$ and $a = 2$. We have $2^{10} = 1024 = 3 \cdot 341 + 1$. Thus, $2^{10} \equiv 1 \pmod{341}$. Then, $2^{341-1} = 2^{340} = (2^{10})^{34} \equiv 1^{34} = 1 \pmod{314}$, but 341 is composite, since $341 = 11 \cdot 31$. That is, Fermat's Little Theorem does not work as a primality test; the composite numbers n that satisfy $a^{n-1} \equiv 1 \pmod n$ are called *Fermat pseudoprime numbers*.

The next two results establish congruences for Catalan numbers modulo certain powers of 2.

Proposition 2.13. *([8], Theorem 4.1) Let C_n be the n-th Catalan number. Then, the following congruences are valid*

$$C_n \equiv \begin{cases} 1 \pmod 8 & \text{if } n = 0 \text{ or } 1 \\ 2 \pmod 8 & \text{if } n = 2^k + 2^{k+1} - 1 \text{ for some } k \geq 0; \\ 4 \pmod 8 & \text{if } n = 2^{k_1} + 2^{k_2} + 2^{k_3} - 1 \text{ for some } k_3 > k_2 > k_1 \geq 0; \\ 5 \pmod 8 & \text{if } n = 2^k - 1 \text{ for some } k \geq 2; \\ 6 \pmod 8 & \text{if } n = 2^{k_1} + 2^{k_2} - 1 \text{ for some } k_2 - 2 \geq k_1 \geq 0; \\ 0 \pmod 8 & \text{otherwise.} \end{cases}$$

Proposition 2.14. *([8], Theorem 6.6) Let $n = 2^k - 1$ for some $k \geq 5$. Then, $C_n \equiv 29 \pmod{64}$.*

Propositions 2.13 and 2.14 will be important for us to establish in the next section certain congruences for Narayana numbers modulo powers of 2.

The next results establish congruences for binomial coefficients modulo powers of a prime p. They will be useful both for defining divisibility rules for Narayana numbers and can be used to determine congruence properties for Catalan numbers. The results was developed by Helou and Terjanian in [5]. Define $w_n = \binom{2n}{n}$ for $n \in \mathbb{N}$.

Proposition 2.15. *(Wolstenholme's theorem) Let $p > 3$ be a prime. Then, $w_p \equiv 2 \pmod{p^3}$.*

Proposition 2.16. *([5], Corollary 4) Let p, q be distinct primes greater than 3. Then, the following holds*

a) $w_{pq} \equiv w_p \pmod{q^3}$ and $w_{pq} \equiv w_q \pmod{p^3}$;

b) $w_{pq} \equiv 2 \pmod{p^3q^3}$ if, and only if $w_p \equiv 2 \pmod{q^3}$ and $w_q \equiv 2 \pmod{p^3}$.

Proposition 2.17. *([5], Proposition 7) For any positive integer k we have that $w_{3^k} \equiv 20 \pmod{3^5}$.*

Based on the results established previously, in the next section we develop some identities involving congruences for the central Narayana numbers modulo certain prime numbers and their powers.

3 Congruences for the central Narayana numbers

In this section, we will explore some results involving the divisibility of Narayana numbers by primes or their powers. In particular, we will work with the central Narayana numbers, which are given, from Proposition 2.4, by the explicit formula

$$\hat{N}(2n+1, n+1) = \frac{1}{2n+1}\binom{2n+1}{n+1}\binom{2n+1}{n} = \frac{1}{2n+1}\binom{2n+1}{n+1}^2. \tag{3.1}$$

In Figure 1 we see the Narayana triangle. The numbers highlighted in the red rectangle are the central Narayana numbers. Note that in the triangle, the numbers are symmetric with respect to the vertical line of the central numbers.

Figure 1: Narayana Triangle

$n=0$						1					
$n=1$					1		1				
$n=2$				1		3		1			
$n=3$			1		6		6		1		
$n=4$		1		10		20		10		1	
$n=5$	1		15		50		50		15		1
$n=6$	1	21		105		175		105		21	1

The central Narayana numbers have a close relationship with the Catalan numbers, as shown in Theorem 3.2. But first, we will present another formula for Narayana numbers. The proof of it is done using an alternative definition for $N(n,k)$ through triangular numbers (for more details see [13], Theorem 1).

Theorem 3.1. *([13], Theorem 1) Let $n \geq 0$ e $0 \leq k \leq n$. The Narayana numbers satisfy the following relation*

$$N(n,k) = \frac{1}{n-k+1}\binom{n}{k}\binom{n+1}{k+1}.$$

Consider $\hat{N}(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$ the formula given by Proposition 2.4. On the other hand, Theorem 3.1 establishes that $N(n,k) = \frac{1}{n-k+1}\binom{n}{k-1}\binom{n+1}{k+1}$ is an another formula for Narayana numbers. Using some binomial properties and further manipulations, it can be proven that $\hat{N}(n+1,k+1) = N(n,k)$. Thus, the formulas presented in Proposition 2.4 and Theorem 3.1 are equivalent.

Consequently, the central Narayana numbers $\hat{N}(2n+1,n+1)$ coincides with $N(2n,n)$, for all $n \geq 1$. From now on, we will use the formula presented in Theorem 3.1 for Narayana numbers.

In the next result, we establish an identity that determines a direct relationship between the central Narayana numbers $N(2n,n)$ and the Catalan numbers.

Theorem 3.2. *([13], Theorem 5) Let n be a non-negative integer. The Narayana numbers satisfy the following identity*

$$N(2n,n) = (2n+1)C_n^2, \tag{3.2}$$

where C_n is the n-th Catalan number. Furthermore, $N(2n,n)$ is a perfect square if, and only if $2n+1$ is a perfect square.

Proof. The proof is done by manipulating the formula for $N(2n, n)$ given by Theorem 3.1 until the desired identity is obtained. ■

In Table 4 we present some Narayana numbers calculated from Identity (3.2) and also highlight some cases where $N(2n, n)$ is a perfect square.

Table 4: Perfect Square Central Narayana Numbers

n	C_n	C_n^2	$2n+1$	$(2n+1)\cdot C_n^2 = N(2n,n)$
0	1	1	1	$1 = 1^2$
1	1	1	3	3
2	2	4	5	20
3	5	25	7	175
4	14	196	9	$1764 = 42^2$
⋮	⋮	⋮	⋮	⋮
12	208012	43268992144	25	$1081724803600 = 1040060^2$
⋮	⋮	⋮	⋮	⋮

Note that, from Identity (3.2) we can establish conditions for the parity of the central Narayana numbers. Since $2n + 1$ is odd for all $n \in \mathbb{N}$, we need to look at the parity of C_n. With this we obtain the next proposition.

Proposition 3.3. *Let $n \geq 1$ be an even integer. Then, $N(2n, n) \equiv 0 \pmod 2$.*

Proof. If n is even, we have by Proposition 2.6 that C_n is even. Consequently, C_n^2 is even. Therefore, $N(2n, n) = (2n + 1)C_n^2$ is even. ■

By Identity (3.2) we have that $N(2n, n)$ is even (or odd) if, and only if, C_n^2 is even (or odd). On the other hand, C_n^2 is even (or odd) if, and only if C_n is even (or odd). Hence, if n is odd, we have, by Corollary 2.8, that C_n is even (or odd) if, and only if, $C_{\frac{n-1}{2}}$ is even (or odd). Thus, we obtain the next result.

Proposition 3.4. *Let $n \geq 1$ be an odd integer. Then,*

$$N(2n, n) \equiv \begin{cases} 0 \pmod 2, & \text{if, and only if, } C_{\frac{n-1}{2}} \text{ is even} \\ 1 \pmod 2, & \text{if, and only if, } C_{\frac{n-1}{2}} \text{ is odd} \end{cases}$$

By Theorem 2.9 we have that C_n is odd if, and only if, n is a Mersenne number. In turn, by Identity (3.2) we have that $N(2n, n)$ is odd if, and only if, C_n is odd. Therefore, $N(2n, n)$ is odd if, and only if, n is a Mersenne number. With this we obtain the next result.

Proposition 3.5. *([13], Theorem 6) Let $n \geq 1$ be an integer. Then, $N(2n, n)$ is odd if, and only if, n is a Mersenne number.*

The next theorem tells us that if p is a prime number and k is any positive integer, then $p \nmid C_{p^k-1}$, where C_n is the n-th Catalan number (see [2], Section 5). To prove this result, Burns uses the theory of finite automata developed by Rowland and Yassawi in [12] to study the behavior of Catalan numbers. The representation of $n = p^k - 1$ in base p consists only of the maximum digit $p - 1$. The automata was constructed so that, when processing this digit repeatedly, the final state is always the constant -1, which implies that $C_{p^k-1} \equiv -1 \pmod{p}$. That is, $p \nmid C_{p^k-1}$.

Theorem 3.6. *For every prime number p and integer $k \geq 1$, it is true that $p \nmid C_{p^k-1}$.*

With Theorem 3.6 in hand, we next establish a new result that tells us that the central Narayana numbers for n of the form $p^k - 1$, with p prime, are not divisible by p.

Theorem 3.7. *Let p be a prime number and $k \geq 1$ be an integer. Then, $p \nmid N(2p^k - 2, p^k - 1)$.*

Proof. It follows from Theorem 3.2 that

$$N(2p^k - 2, p^k - 1) = (2p^k - 1) \cdot C^2_{p^k-1}.$$

Now, by Theorem 3.6 we have that $p \nmid C_{p^k-1}$, which implies $p \nmid C^2_{p^k-1}$. On the other hand, $2p^k - 1 \equiv -1 \equiv p - 1 \pmod{p}$. Thus, $p \nmid 2p^k - 1$. Therefore, $p \nmid N(2n, n)$ for $n = p^k - 1$. ■

Bóna and Sagan investigate in [1] divisibility properties of the Narayana numbers $N(n, k)$ by a given prime p. To establish their main result, they utilize the base-p expansion of n and k as a key tool, alongside Kummer's Theorem, which provides a useful way of determining the p-adic valuation of binomial coefficients. This approach is particularly relevant since Narayana numbers can be expressed in terms of binomial coefficients. The authors thereby establish general divisibility criteria for $N(n, k)$ by a prime p, depending on whether p divides n or not.

Our work, however, focuses specifically on the *central* Narayana numbers and determines divisibility criteria for this family of numbers modulo a given prime p and its powers. Similar to the approach of Bóna and Sagan, we also exploit the base-p expansion of n to establish new congruences, as can be seen in the next result.

Theorem 3.8. *Let $p > 3$ be a prime number and $k \in \mathbb{N}$. If n is of the form $\dfrac{p^{k+1} - 1}{p - 1}$ then $p \nmid N(2n, n)$.*

Proof. By Theorem 3.2 we have that $N(2n,n) = (2n+1)C_n^2$, where C_n is the $n-$th Catalan number. Note that, $n = 1+p+\cdots+p^k$. Thus, $2n+1 = 2p(1+p+\cdots+p^{k-1})+3$. Since $p > 3$ then $p \nmid (2n+1)$. On the other hand, $C_n = \dfrac{1}{n+1}\binom{2n}{n}$. As $n+1 = p(1+p+\cdots+p^{k-1})+2$, $p \nmid (n+1)$, $p \nmid 2$ and $n+1 \equiv 2 \pmod{p}$, then $\dfrac{1}{n+1} \equiv \dfrac{1}{2} \pmod{p}$. Now, if $a = \sum_{j=0}^{m} a_j p^j$ and $b = \sum_{j=0}^{m} b_j p^j$ we have, by Lucas' Theorem, that

$$\binom{a}{b} \equiv \prod_{j=0}^{m} \binom{a_j}{b_j} \pmod{p}. \tag{3.3}$$

Taking $a = 2n$ and $b = n$ in (3.3) it follows that $\binom{2n}{n} \equiv \prod_{j=0}^{k} \binom{2}{1} \pmod{p}$. That is, $\binom{2n}{n} \equiv 2^{k+1} \pmod{p}$. Hence, when $n = \dfrac{p^{k+1}-1}{p-1}$ we have that $C_n \equiv 2^k \pmod{p}$. Thus, $p \nmid C_n$ and, consequently, $p \nmid C_n^2$. Therefore, $p \nmid N(2n,n)$. ■

Note that, if $p = 2$ then $n = 2^{k+1} - 1$ and we fall back to Theorem 3.7 and, as we know, $2 \nmid N(2n,n)$. As immediate consequences of the proof of Theorem 3.8, we obtain the next result.

Corollary 3.9. Let $p > 3$ be a prime number and $k \geq 0$ an integer number. If n is of the form $\dfrac{p^{k+1}-1}{p-1}$ then it holds that

a) $C_n \equiv 2^k \pmod{p}$;

b) $N(2n,n) \equiv 3 \cdot 2^{2k} \pmod{p}$;

c) $p \mid N(2n,n)$ if, and only if, $p = 3$.

Proof. Item *a)* follows directly from Lucas' Theorem. As $2n+1 \equiv 3 \pmod{p}$ and $C_n \equiv 2^k \pmod{p}$, then $N(2n,n) \equiv 3 \cdot 2^{2k} \pmod{p}$. Again, $2n+1 \equiv 3 \pmod{p}$, so, $3 \mid (2n+1)$, consequently, $3 \mid N(2n,n)$ when $n = \dfrac{3^{k+1}-1}{2}$. ■

In contrast to Theorem 3.8, we next establish a result in which we explore the divisibility of the numbers $N(2n,n)$ when n is written in base p, with p prime, but with not all coefficients equal to 1. We establish a condition on these coefficients such that p does not divide $N(2n,n)$.

Theorem 3.10. *Let $p > 3$ be a prime, $k \in \mathbb{N}$ and $n = \sum_{m=0}^{k} c_m p^m$. If c_m is an integer such that $0 \leq c_m < \frac{p-1}{2}$ for all $0 < m \leq k$ then $p \nmid N(2n, n)$.*

Proof. By Theorem 3.2 we have $N(2n, n) = (2n+1)C_n^2$. Note that $2n+1 = (2c_0+1) + p\sum_{m=1}^{k} c_m p^{m-1}$. As $2c_0 + 1 < p$ then $p \nmid (2n + 1)$.

Now, by Lucas' Theorem we have that $\binom{2n}{n} \equiv \prod_{m=0}^{k} \binom{2c_m}{c_m} \pmod{p}$, as $2c_m < p - 1 < p$ then p does not divide $\binom{2c_m}{c_m}$ for each m. Therefore, $p \nmid C_n^2$. Consequently, $p \nmid N(2n, n)$. ■

Note that in Theorem 3.10 we have that $\frac{p-1}{2}$ is always an integer, since every $p > 2$ is odd, then $p - 1$ is even, so the condition for c_m is well established.

Now observe that, if k is an odd positive integer and $p > 2$ is prime, then kp is odd, so $n = kp-1$ is even. Therefore, $n = \frac{kp-1}{2}$ is an integer. In this way, $2n+1 = kp$ and, consequently, $p \mid (2n+1)$. As $N(2n, n) = (2n+1)C_n^2$ by Theorem 3.2, then $p \mid N(2n, n)$. From this discussion, the next result follows.

Proposition 3.11. *Let $k \in \mathbb{N}$ be odd, $p > 2$ be prime and $n = \frac{kp-1}{2}$. Then, $N(2n, n) \equiv 0 \pmod{p}$.*

Note that $2n + 1 = kp$, so we can say that k also divides $N(2n, n)$. However, we are interested in the prime divisors of $N(2n, n)$ and their powers.

The next theorem is an interesting result that starts from Fermat's Little Theorem and allows establishing a divisibility criterion for the central Narayana numbers.

Theorem 3.12. *Let $p > 2$ be a prime and $a \in \mathbb{N}$ even such that $p \nmid a$. If $n = \frac{a^{p-1}-2}{2}$ then $N(2n, n) \equiv 0 \pmod{p}$.*

Proof. Since $p \nmid a$ we have, by Fermat's Little Theorem, that $a^{p-1} \equiv 1 \pmod{p}$. Thus, $2n + 1 \equiv 0 \pmod{p}$. Since $N(2n, n) = (2n + 1)C_n^2$, then $p \mid N(2n, n)$. ■

Note that we take a as an even natural number in Theorem 3.12 because, if a is even, then a^m is even, for any natural number m. Then, $a^m - 2$ remains even, and thus, $\frac{a^m-2}{2}$ is an integer. Furthermore, to maintain one of the conditions of Fermat's

Little Theorem, we need $p \nmid a$, and as a is even, p cannot be equal to 2, which is why we require $p > 2$ prime.

Now, as a consequence of Proposition 2.13 and Theorem 3.2 we establish the following result, which provides congruences for $N(2n, n)$ modulo sum of powers of 2.

Theorem 3.13. *Let $N(2n, n)$ be the n-th central Narayana number. Then, the following congruences are valid*

$$N(2n,n) \equiv \begin{cases} 4 \pmod 8 & \text{if } n = 2^k + 2^{k+1} - 1 \text{ for some } k \geq 0; \\ 0 \pmod 8 & \text{if } n = 2^{k_1} + 2^{k_2} + 2^{k_3} - 1 \text{ for some } k_3 > k_2 > k_1 \geq 0; \\ 7 \pmod 8 & \text{if } n = 2^k - 1 \text{ for some } k \geq 2; \\ 4 \pmod 8 & \text{if } n = 2^{k_1} + 2^{k_2} - 1 \text{ for some } k_2 - 2 \geq k_1 \geq 0. \end{cases}$$

Proof. We will prove the case where $n = 2^k - 1$ for some $k \geq 2$, the other cases follow analogously. By Proposition 2.13 we have that $C_n \equiv 5 \pmod 8$, so, $C_n^2 \equiv 1 \pmod 8$. On the other hand, $2n + 1 = 2^{k+1} - 1$, as $k \geq 2$, then $k + 1 \geq 3$, therefore, $8 \mid 2^{k+1}$ and, thus, $2n + 1 \equiv -1 \equiv 7 \pmod 8$. Therefore, $N(2n, n) \equiv 7 \pmod 8$. ∎

Note that, if $N(2n, n) \equiv b \pmod 8$, then there exists $k \in \mathbb{Z}$ such that $N(2n, n) = 8k + b$. Thus, if $m \mid 8$, we only need to find the remainder of the division of b by m to generate new congruences. As 8 is divisible by 2 and 4, we obtain, from Theorem 3.13, the next result.

Theorem 3.14. *Let $N(2n, n)$ be the n-th central Narayana number. Then, the following congruences are valid*

$$N(2n,n) \equiv \begin{cases} 0 \pmod 2 & \text{if } n = 2^k + 2^{k+1} - 1 \text{ for some } k \geq 0; \\ 0 \pmod 2 & \text{if } n = 2^{k_1} + 2^{k_2} + 2^{k_3} - 1 \text{ for some } k_3 > k_2 > k_1 \geq 0; \\ 1 \pmod 2 & \text{if } n = 2^k - 1 \text{ for some } k \geq 2; \\ 0 \pmod 2 & \text{if } n = 2^{k_1} + 2^{k_2} - 1 \text{ for some } k_2 - 2 \geq k_1 \geq 0. \end{cases}$$

$$N(2n,n) \equiv \begin{cases} 0 \pmod 4 & \text{if } n = 2^k + 2^{k+1} - 1 \text{ for some } k \geq 0; \\ 0 \pmod 4 & \text{if } n = 2^{k_1} + 2^{k_2} + 2^{k_3} - 1 \text{ for some } k_3 > k_2 > k_1 \geq 0; \\ 3 \pmod 4 & \text{if } n = 2^k - 1 \text{ for some } k \geq 2; \\ 0 \pmod 4 & \text{if } n = 2^{k_1} + 2^{k_2} - 1 \text{ for some } k_2 - 2 \geq k_1 \geq 0. \end{cases}$$

The next result follows as a consequence of Proposition 2.14 and provides a congruence for $N(2n, n)$ modulo 64 when n is a Mersenne number.

Proposition 3.15. *If $n = 2^k - 1$ for $k \geq 5$, then $N(2n, n) \equiv 55 \pmod{64}$.*

Proof. According to Proposition 2.14 we have that $C_n \equiv 29 \pmod{64}$, given the hypotheses of the problem. Thus, $C_n^2 \equiv 9 \pmod{64}$. In turn, as $k \geq 5$, then $k + 1 \geq 6$, thus $64 \mid 2^{k+1}$. In turn, $2n + 1 \equiv -1 \equiv 63 \pmod{64}$. Therefore, $N(2n, n) \equiv 55 \pmod{64}$. ■

Similarly to what we did in Theorem 3.14, we look at the divisors d of 64 and the remainder of the division of 55 by d, thus obtaining new congruences for the central Narayana numbers modulo powers of 2.

Corollary 3.16. If $n = 2^k - 1$ for $k \geq 5$, then the following congruences are valid

$$N(2n, n) \equiv \begin{cases} 1 & \pmod 2; \\ 3 & \pmod 4; \\ 7 & \pmod 8; \\ 7 & \pmod{16}; \\ 23 & \pmod{32}. \end{cases}$$

Proof. Since $N(2n, n) \equiv 55 \pmod{64}$, there exists an integer k such that $N(2n, n) = 64k + 55$. For $1 \leq m \leq 5$ we have that $2^m \mid 64$. Then, $N(2n, n) \equiv l \pmod{2^m}$, where $l = 55 \pmod{2^m}$. ■

Let $p > 3$ be a prime number. By definition we have that $C_p = \dfrac{1}{p+1}\binom{2p}{p}$. On the other hand, according to Wolstenholme's Theorem we have that if $p > 3$ is prime, then $\binom{2p}{p} \equiv 2 \pmod{p^3}$. Note that the multiplicative inverse of $1 + p$ is $1 - p + p^2$, since

$$(1 + p)(1 - p + p^2) = 1 + p^3 \equiv 1 \pmod{p^3}.$$

Thus, $C_p \equiv 2(1 - p + p^2) \pmod{p^3}$. Therefore, $C_p^2 \equiv 4(1 - 2p + 3p^2) \pmod{p^3}$. By Theorem 3.2 we have that $N(2p, p) = (2p + 1)C_p^2$. It follows that $N(2p, p) \equiv 4 - 4p^2 \pmod{p^3}$. From this discussion, the next result is obtained.

Theorem 3.17. *Let $p > 3$ be prime. Then, $N(2p, p) \equiv 4 - 4p^2 \pmod{p^3}$.*

Example 3.18. Let $p = 5$. Then, $2p + 1 = 11$ and $C_5 = 42$. So, $N(10, 5) = 11 \cdot 42^2 = 19404$. Thus, $N(10, 5) \equiv 29 \pmod{5^3}$.

On the other hand, $4 - 4p^2 = 4 - 4 \cdot 25 = -96$ and $-96 \equiv 29 \pmod{5^3}$. Therefore, $N(10, 5) \equiv -96 \equiv 29 \pmod{5^3}$.

In Theorem 3.19, we will explore the divisibility of the central Narayana numbers when n is equal to the product of two distinct primes $p, q \geq 5$ modulo p^3 or q^3.

Theorem 3.19. *Let $p, q \geq 5$ be distinct primes. Then, it holds that*

$$N(2pq, pq) \equiv \begin{cases} (1-p^2q^2)\binom{2q}{q}^2 & (\text{mod } p^3) \\ (1-p^2q^2)\binom{2p}{p}^2 & (\text{mod } q^3) \end{cases}.$$

Proof. According to item a) of Proposition 2.16, we have that $\binom{2pq}{pq} \equiv \binom{2q}{q} \pmod{p^3}$ and $\binom{2pq}{pq} \equiv \binom{2p}{p} \pmod{q^3}$. We will prove the identity modulo p^3, the other follows analogously. As $C_{pq} = \frac{1}{1+pq}\binom{2pq}{pq}$, then $C_{pq} \equiv \frac{1}{1+pq}\binom{2q}{q} \pmod{p^3}$. In turn, $\frac{1}{1+pq} = 1-pq+p^2q^2 \pmod{p^3}$, since $(1+pq)(1-pq+p^2q^2) = 1+p^3q^3 \equiv 1 \pmod{p^3}$. Then, $C_{pq} \equiv (1-pq+p^2q^2)\binom{2q}{q} \pmod{p^3}$. After certain manipulations and congruences we obtain $C_{pq}^2 \equiv (1-2pq+3p^2q^2)\binom{2q}{q}^2 \pmod{p^3}$. As $(2pq+1)(1-2pq+3p^2q^2) \equiv (1-p^2q^2) \pmod{p^3}$, then $N(2pq, pq) \equiv (1-p^2q^2)\binom{2q}{q}^2 \pmod{p^3}$. ∎

Note that, according to item b) of Proposition 2.16, if $w_p \equiv 2 \pmod{q^3}$ and $w_q \equiv 2 \pmod{p^3}$ then $w_{pq} \equiv 2 \pmod{p^3q^3}$. From there, in the same way as we did in the previous demonstration, we have that $C_{pq}^2 \equiv 4(1-2pq+3p^2q^2) \pmod{p^3q^3}$. Performing $(2pq+1)(1-2pq+3p^2q^2)$ we obtain that $N(2pq, pq) \equiv 4(1-p^2q^2) \pmod{p^3q^3}$. Hence, the next theorem follows.

Theorem 3.20. *Let $p, q \geq 5$ be distinct primes and $w_n = \binom{2n}{n}$, $n \in \mathbb{N}$. If $w_p \equiv 2 \pmod{q^3}$ and $w_q \equiv 2 \pmod{p^3}$, then $N(2pq, pq) \equiv 4(1-p^2q^2) \pmod{p^3q^3}$.*

In the next result we will explore the central $N(2n, n)$ when $n = 3^k$ for some integer $k \geq 1$.

Theorem 3.21. *If $n = 3^k$ for some $k \geq 1$, then* $N(2n, n) \equiv \begin{cases} 175 \pmod{3^5}, & k = 1; \\ 76 \pmod{3^5}, & k = 2; \\ 157 \pmod{3^5}, & k \geq 3. \end{cases}$

Proof. According to Proposition 2.17 we have that if $n = 3^k$, then $\binom{2n}{n} \equiv 20 \pmod{3^5}$. Thus, $C_{3^k} \equiv \dfrac{20}{1+3^k} \pmod{3^5}$. Note that the multiplicative inverses of $\dfrac{1}{1+3^k}$, for $k = 1, 2, 3, 4$ are, respectively, $61, 73, 217$ and 163. On the other hand, for $k \geq 5$ we have that $\dfrac{1}{1+3^k} \equiv 1 \pmod{3^5}$, since $3^5 \mid 3^k$ for all $k \geq 5$. Thus,

$$C_{3^k} \equiv \begin{cases} 5 \pmod{3^5}, & k = 1; \\ 2 \pmod{3^5}, & k = 2; \\ 209 \pmod{3^5}, & k = 3; \\ 101 \pmod{3^5}, & k = 4; \\ 20 \pmod{3^5}, & k \geq 5. \end{cases} \Rightarrow C_{3^k}^2 \equiv \begin{cases} 25 \pmod{3^5}, & k = 1; \\ 4 \pmod{3^5}, & k = 2; \\ 184 \pmod{3^5}, & k = 3; \\ 238 \pmod{3^5}, & k = 4; \\ 157 \pmod{3^5}, & k \geq 5. \end{cases}$$

Furthermore, since $2n + 1 = 2 \cdot 3^k + 1$, then

$$2 \cdot 3^k + 1 \equiv \begin{cases} 7 \pmod{3^5}, & k = 1; \\ 19 \pmod{3^5}, & k = 2; \\ 55 \pmod{3^5}, & k = 3; \\ 163 \pmod{3^5}, & k = 4; \\ 1 \pmod{3^5}, & k \geq 5. \end{cases}$$

Therefore, since $N(2 \cdot 3^k, 3^k) = (2 \cdot 3^k + 1)C_{3^k}^2$, then

$$N(2n, n) \equiv \begin{cases} 175 \pmod{3^5}, & k = 1; \\ 76 \pmod{3^5}, & k = 2; \\ 157 \pmod{3^5}, & k = 3; \\ 157 \pmod{3^5}, & k = 4; \\ 157 \pmod{3^5}, & k \geq 5. \end{cases}$$

■

In this section, we developed new identities that arise from congruences for Catalan numbers, which is possible due to the direct relationship that exists between the central Narayana numbers and the sequence C_n, as shown in Theorem 3.2. We established divisibility criteria for $N(2n, n)$ when divided by a prime p and its powers.

4 Conclusions

In this work, we explored divisibility properties for Catalan numbers and also classic results from the literature. From this, we developed new congruences for the central

Narayana numbers, which are a special family of numbers that appear as the central line in the Narayana triangle.

Our objective focused on looking at divisibility conditions for $N(2n, n)$ when divided by a prime p and its powers; in particular cases, we looked at powers of 2 and 3. We also looked at the case where n is a Mersenne number. We believe that the presented results are new in the literature and we hope that the article can contribute to future works.

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